**Comprehensive Exam in Algebra June 2006** **NAME**___________________________________

Do 10 problems, with AT LEAST 2 PROBLEMS FROM EACH SECTION. That is, do at least two problems numbered Gx, two problems numbered Rx, two problems numbered Fx and two problems numbered LAx.

Do each problem on the sheet for that problem. Write your name on each page. Show all details and quote properly any theorems that you use. All problems are worth 10 points. We prefer complete solutions of a few problems to many partial solutions.

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G1. Show that all groups of order 825 are nonsimple.

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G2. Let $G$ be a finite group and let $S$ be a Sylow $p$-subgroup of $G$. Call an element a ***p*-element** if it has order a positive power of $p$. If $S$ is normal in $G$, prove that $S$ is the set of all $p$-elements of $G$, along with the identity element.

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G3. Let $\pi = \{p_1, \ldots, p_m\}$ be a nonempty set of primes. A **$\pi$-group** is a group whose order $n$ has the property that all primes dividing $n$ lie in the set $\pi$. For example, a group of order $2^3 \cdot 5 \cdot 7^2$ is a $\{2, 5, 7\}$-group. Let $G$ be a finite group and let $H$ and $K$ be normal subgroups such that $G/H$ and $G/K$ are $\pi$-groups. Prove that $G/(H \cap K)$ is also a $\pi$-group. *Hint*: Use the isomorphism theorems.

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R4. In the ring $\mathbb{Z}$, find a single generator for each of the following ideals $\langle a \rangle \cap \langle b \rangle$, $\langle a \rangle \langle b \rangle$ and $\langle a \rangle + \langle b \rangle$

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R5. Let $\mathcal{C} = \{P_u \mid u \in U\}$ be a chain (totally ordered family under inclusion) of ideals in a commutative ring $R$ with identity. Prove that the union $U = \bigcup \mathcal{C}$ is an ideal of $R$.

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R6. Prove that a commutative ring $S$ with identity is a field if and only if $S$ has no nonzero proper ideals.

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R7. Let $R$ be a ring with a unique maximal ideal $M$. Show that $M$ is the set of nonunits of $R$.

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F8. If $F < E$ is a field extension of finite degree, prove that $E$ is algebraic over $F$.

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F9. Let $F$ be a field. Show that a polynomial $f(x) \in F[x]$ has no multiple roots if and only if $f(x)$ and its derivative $f'(x)$ are relatively prime. *Hint*: You may assume that $f$ has a splitting field $E$ containing $F$.

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L10. Let $\dim(V) < \infty$ and suppose that $\tau$ is a linear operator on $V$ satisfying $\tau^2 = 0$. Show that $2\text{rk}(\tau) \leq \dim(V)$.

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L11. Show that if $\tau$ and $\sigma$ are invertible linear operators on a vector space $V$ then $\tau\sigma$ and $\sigma\tau$ have the same eigenvalues.

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L12. Let $S$ be a subspace of a finite-dimensional inner product space $V$. Prove that each coset in $V/S$ contains *exactly one* vector that is orthogonal to $S$.