Comprehensive exam – June 2004

Do each problem on a separate page. Write your name on every page.

You have to justify your answers.

1. $y = (y_1, y_2)$ is a function if x, defined by a system of equations

$$\begin{cases} xy_1^2 + y_2 - y_1 y_2 = 1 \\ x^2 + y_1^2 y_2 = 2. \end{cases}.$$

Compute dy_1/dx at the point where $x = y_1 = y_2 = 1$. Justify the existence of this derivative.

2. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$, and assume that this series converges whenever $0 \le x \le 2$.

(a) Prove that the series $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = -3/2$.

(b) Prove that there exists a constant C such that $|a_n| < C(2/3)^n$ for any $n \ge 0$.

(c) Prove that there exists a constant C' such that $|f^{(n)}(1)| \le C' 2^n n!$ for any $n \ge 0$.

3. (a) Define a contractive mapping. State the Contractive Mapping Principle.

(b) For $f \in C([1,2])$, define $Tf \in C([1,2])$ by setting

$$Tf(x) = 2x + 1 + \int_1^x \frac{f(t)}{t}\,dt$$

for $x \in [1,2]$. Prove that T is a contractive mapping on $C([1,2])$.

(c) Use the Contractive Mapping Principle to prove that the system of equations

$$\begin{cases} xf'(x) = f(x) + 2x \\ f(1) = 3 \end{cases}$$

has a unique solution on $[1,2]$.

4. State and prove the Intermediate Value Theorem (in one variable).

5. Suppose A is a subset of a complete metric space (M, d), and f is a uniformly continuous M-valued function, defined on A. Prove that there exists a uniformly continuous function $g : \overline{A} \to M$ such that $g|_A = f$.

6. Suppose $f : [a,b] \to \mathbb{R}$ is Riemann integrable on the interval $[a,b]$, and assume that there exists a positive constant c such that $f(x) \ge c$ for all $x \in [a,b]$. Prove that $1/f$ is Riemann integrable.

7. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ define $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$. Suppose f maps $\mathbb{R}^n$ onto $\mathbb{R}^n$ in such a way that $|f(x) - f(y)| \ge |x - y|$ for any $x, y \in \mathbb{R}^n$. Suppose A is an open subset of $\mathbb{R}^n$. Prove that $f(A)$ is open.

8. Suppose U is an open subset of $\mathbb{R}$, containing a point x_0. f and g are real-valued functions, defined on U, such that g is continuous, f is differentiable, and $f(x_0) = 0$. Prove that the product fg is differentiable at x_0.

9. (a) State Heine-Borel Theorem for subsets of $\mathbb{R}$.

(b) Construct a sequence $(x_n)_{n \in \mathbb{N}}$ of real numbers as follows: set $x_0 = 1$, and let $x_{n+1} = x_n + e^{-x_n} - 1$ for $n \ge 0$. Does the sequence (x_n) converge? If it does, compute $\lim_{n\to\infty} x_n$.