Advisory exam. September 16, 2004

DO EACH PROBLEM ON A SEPARATE PAGE. WRITE YOUR NAME AND STUDENT ID NUMBER ON EVERY PAGE. YOU HAVE TO JUSTIFY YOUR ANSWERS, USING COMPLETE SENTENCES AND PROPER GRAMMAR.

1. We say that a subset E of $\mathbb{R}^n$ has the *Bolzano-Weierstrass property* if any sequence of elements of E has a subsequence which converges to an element of E. E is said to have the *Heine-Borel property* if every open cover of E has a finite subcover.

It is known that a set E has the Heine-Borel property if and only if it has the Bolzano-Weierstrass property. Prove either the "if," or the "only if" implication (indicate precisely which implication you are proving).

2. Prove or disprove: if $f : \mathbb{R}^2 \to \mathbb{R}$ has the property that $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ both exist, then f is continuous at $(0,0)$.

3. Suppose $f : [a,b] \to \mathbb{R}$ is a monotone function. Prove that f is Riemann integrable on $[a,b]$.

4. Suppose that $(f_1, f_2, \ldots)$ is a sequence of real-valued uniformly continuous functions on $\mathbb{R}$, which converges uniformly to a function $g : \mathbb{R} \to \mathbb{R}$. Prove or disprove: g is also uniformly continuous on $\mathbb{R}$.

5. Consider two subsets of $\mathbb{R}^2$:

$S_1 = \{(x,y) \mid x^3 - 3xy + y^3 = 1\}$, $S_2 = \{(x,y) \mid x^4 + 2x^2y^2 + 4x + 4y = 4\}$.

(a) Prove that S_1 and S_2 both contain the point $(0,1)$, and that they are both C^1 curves in a neighborhood of $(0,1)$.

(b) Prove that the tangent lines to S_1 and S_2 at $(0,1)$ are orthogonal.

6. (a) For what real values of x does the series

$$f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}$$

converge?

(b) Compute $f'(1/3)$. Justify your answer.

7. Define a real-valued function f as follows: if $x \in \mathbb{R}$ is irrational, set $f(x) = 0$. If x is rational and non-zero, represent it as p/q, with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ having no common factors, and let $f(x) = 2^{-q}$. Set $f(0) = 1$.

Find all the points x where f is continuous.

8. Suppose a real-valued function f is twice continuously differentiable in a neighborhood of a point $x \in \mathbb{R}$. Prove that

$$f''(x) = \lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}.$$

9. (a) Define the lim sup and lim inf of a sequence of real numbers.

(b) Give an example of a sequence $(a_n)_{n \in \mathbb{N}}$ of real numbers for which $\liminf a_n = -1$, and $\limsup a_n = 3$. Explain, using the definition given in part (a), why this sequence has the desired properties.