Comprehensive Exam in Real Analysis Fall 2006

Thursday September 14, 2006 9:00-11:30am

Exam Packet #

Name (please print): ______________________________

Student ID: ___ ___ ___ ___ ___ ___ ___ ___

INSTRUCTIONS

(1) The examination is divided into three sections to simplify the paper handling:
Part A: ('Green Pages') Problems 1, 2 and 3
Part B: ('Pink Pages') Problems 4, 5 and 6
Part C: ('Yellow Pages') Problems 7, 8 and 9
Before going on, please print your name at the indicated places on each part.

(2) Each of the nine problems is given the same credit, so don't get bogged down on a single problem.

(3) Write your answer to each question on the sheet containing that question; you may use the back of that sheet if you need to. If you need even more room for a given question, ask the exam proctor for extra paper and attach it to the original packet.

(4) Provide complete answers, written in complete sentences.

(5) The examination is 'completely closed book': No books, notes, calculators, PDAs, etc.

(6) Please do not write in the 'Score Boxes' on the exam pages.

(7) Enjoy the exam!

NOTATIONS USED IN THIS EXAM

- The symbol $\mathbb{N}$ denotes the set of all positive integers

- If n is in $\mathbb{N}$ then $\mathbb{R}^n$ denotes the set of all ordered n-tuples of real numbers; in particular, $\mathbb{R} = \mathbb{R}^1$ denotes the set of all real numbers.

- Elements of $\mathbb{R}^n$ are sometimes denoted by boldface letters; for example, $\mathbf{u} = (u_1, \ldots u_n)$, where the real numbers $u_1, \ldots\ u_n$ are the *components* of the vector $\mathbf{u}$. Sometimes the vector $\mathbf{u} = (u_1, \ldots u_n)$ is thought of as a column vector and written $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$.

- If $\mathbf{u} = (u_1, \ldots u_n)$ is an element of $\mathbb{R}^n$, then the **Euclidean norm** or **length** of $\mathbf{u}$ is the number $||\mathbf{u}||_2 = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$.

(1) Analyze the convergence behavior of each of the following infinite series. (By the command 'Analyze the convergence behavior' of a series $\sum a_n$, it is meant: (i) Determine whether the series converges or diverges; (ii) if it converges, determine whether it converges absolutely or conditionally; (iii) if it diverges, determine whether or not the sequence a_n converges to 0.)

$$(a)\ \frac{1}{2} - \frac{1}{3} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{2^3} - \frac{1}{3^3} + \frac{1}{2^4} - \frac{1}{3^4} + \cdots + \frac{1}{2^n} - \frac{1}{3^n} + \cdots$$

$$(b)\ \frac{1}{2(\ln 2)^{3/2}} + \frac{1}{3(\ln 3)^{3/2}} + \frac{1}{4(\ln 4)^{3/2}} + \cdots + \frac{1}{n(\ln n)^{3/2}} + \cdots$$

Justify your answers.

(2) Throughout this problem let $f : \mathbb{R}^n \to \mathbb{R}$ be function with continuous first partials at each point of $\mathbb{R}^n$.

(a) Prove that f is continuous at each point of $\mathbb{R}^n$.

(b) Suppose that f has exactly one critical point, namely at a certain point $\mathbf{x}_0 \in \mathbb{R}^n$. Suppose further that $\lim_{|\mathbf{x}| \to +\infty} f(\mathbf{x}) = +\infty$.

Prove or Disprove: The function f takes on its global minimum at the point $\mathbf{x}_0$.

(3) Prove or Disprove: If $f : [a,b] \to \mathbb{R}$ is a continuous real-valued function such that $\int_a^b |f(x)|\, dx = 0$, then $f(x) = 0$ for all $x \in [a,b]$.

(4) Let f be a real-valued function with domain $\mathbb{R}$, and let A and y be elements of $\mathbb{R}$. Prove that if $\limsup_{x\to y} f(x) = A$ then there is a sequence (x_n) in $\mathbb{R}$ such that

$$y = \lim_{n\to\infty} x_n \text{ and } A = \lim_{n\to\infty} f(x_n).$$

(5) Reminder: Suppose that (a_n) is an infinite sequence of *positive* real numbers, and for each positive integer n let p_n denote the finite product $a_1 \cdot a_2 \cdot \ldots \cdot a_n$. Then the 'infinite product' $a_1 \cdot a_2 \cdot \ldots \cdot a_n \cdot \ldots$ is said to be **convergent** if the sequence (p_n) converges to a *nonzero* finite value p. When this happens one writes

$$a_1 \cdot a_2 \cdot \ldots \cdot a_n \cdot \ldots = p; \text{ or, more briefly, } \prod_{n=1}^{\infty} a_n = p.$$

One then calls p the **value** of this infinite product.

Problem:

(a) Show that the infinite product $\displaystyle\prod_{n=1}^{\infty} \left(1 + \frac{1}{n^2}\right)$ is convergent.

(b) Estimate the value of the infinite product in Part (a) correct to one decimal place. Justify your estimate.

(6) Prove, from the definition of 'Riemann integrable', that if f and g are Riemann integrable on $[a, b]$, then $f \cdot g$ is also Riemann integrable on $[a, b]$.

(7) Consider the sequence $(x_0, x_1, x_2, \ldots x_n, \ldots)$ defined recursively by the rule

$$\begin{cases} x_0 & = & 1 \\ x_{n+1} & = & \tanh(x_n) \text{ for } n \, = \, 0, 1, 2, \ldots \end{cases}$$

Problem: Determine the 'convergence' properties of this sequence. That is, determine whether it is convergent; if it is convergent, determine its limit; if it is not convergent, determine how it diverges (e.g., to one of the infinities? oscillatory?) Justify your answers.

Reminder: The formula for the 'hyperbolic tangent function' $\tanh(x)$ can be written

$$\tanh(x) \, = \, \frac{\sinh x}{\cosh x} \, = \, \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

(8) Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$, $n = 1, 2, 3 \ldots$, given by the formula

$$f_n(x) = \frac{n}{n + x^2}, \text{ for all } x \in \mathbb{R}.$$

(a) Show that this sequence converges pointwise on $\mathbb{R}$ to some limit function $g : \mathbb{R} \to \mathbb{R}$, and determine the formula for g.

(b) Determine whether the sequence converges *uniformly* to the function g obtained in Part (a) above.

Justify your answers.

(9) Let $f : \overline{B}_1 \rightarrow (0, \infty)$ be a continuous function, with values in the set of positive real numbers, defined on the standard closed unit ball $\overline{B}_1$ in $\mathbb{R}^n$.

Problem: Determine the convergence properties of the infinite series $\sum_{n=1}^{\infty} e^{-nf(\mathbf{x})}$. That is, determine whether the series converges pointwise on $\overline{B}_1$; and if it does, then also determine whether the convergence is uniform, and compute the formula for the function to which the series converges.

Justify your answers.