Comprehensive Exam in Real Analysis Spring 2006

Exam Packet #

Name (please print):

Student ID:

INSTRUCTIONS

(1) The examination is divided into three sections to simplify the paper handling:
Part A: ('Green Pages') Problems 1, 2 and 3
Part B: ('Pink Pages') Problems 4, 5 and 6
Part C: ('Yellow Pages') Problems 7, 8 and 9
Please print your name at the indicated places on each part.

(2) Each of the nine problems is given the same credit, so don't get bogged down on a single problem.

(3) Write your answer to each question on the sheet containing that question; you may use the back of that sheet if you need to. If you need even more room for a given question, ask the exam proctor for extra paper and attach it to the original packet.

(4) Provide complete answers, written in complete sentences.

(5) The examination is 'completely closed book': No books, notes, calculators, PDAs, etc.

(6) Please do not write in the 'Score Boxes' on the exam pages.

(7) Enjoy the exam!

NOTATIONS USED IN THIS EXAM

- The symbol $\mathbb{N}$ denotes the set of all positive integers

- If n is in $\mathbb{N}$ then $\mathbb{R}^n$ denotes the set of all ordered n-tuples of real numbers; in particular, $\mathbb{R} = \mathbb{R}^1$ denotes the set of all real numbers.

- Elements of $\mathbb{R}^n$ are sometimes denoted by boldface letters; for example, $\mathbf{u} = (u_1, \ldots u_n)$, where the real numbers $u_1, \ldots\ u_n$ are the *components* of the vector $\mathbf{u}$. Sometimes the vector $\mathbf{u} = (u_1, \ldots u_n)$ is thought of as a column vector and written $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$.

- If $\mathbf{u} = (u_1, \ldots u_n)$ is an element of $\mathbb{R}^n$, then the **Euclidean norm** or **length** of $\mathbf{u}$ is the number $||\mathbf{u}||_2 = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$.

(1) Prove or Disprove: The Euclidean plane $\mathbb{R}^2$ can be expressed as the countable union of straight lines.

(2) (a) Determine whether the infinite series $\sum_{n=1}^{\infty} \sin(n^{-2}) \cos(n^{-1})$ is convergent.

(b) Suppose that $f \in \mathrm{BUC}^2(\mathbb{R}, \mathbb{R})$; that is, f, f' and f'' are all bounded and uniformly continuous real-valued functions on $\mathbb{R}$. Assume further that $f(0) = f'(0) = 0$.

Problem: Analyze the convergence of the series $\sum_{n=1}^{\infty} f\left(\frac{x}{n}\right)$.

Note: In Part (b) the requirement to 'analyze the convergence' means that you should determine for which values of x the series is convergent; at values of x where the series converges, determine whether the convergence is absolute; and determine whether the convergence is uniform.

Needless to say, in both parts of this problem you should prove your claims.

(3) (a) Carefully state one of the following theorems:

(i) The Bolzano-Weierstrass Theorem for sequences of real numbers; or

(ii) The Heine-Borel Theorem for the real numbers.

Note: If you elect to do (ii), and your statement of the theorem involves the phrase 'Heine-Borel Property', be sure to carefully state what that property is.

(b) Let x_0 and y_0 be real numbers such that $x_0 > y_0 > 0$. Consider the numeric sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ defined recursively by the rule

$$\left\{\begin{array}{lcl} x_{n+1} & = & \dfrac{x_n + y_n}{2} \\ & & \\ y_{n+1} & = & \sqrt{x_n y_n} \end{array}\right. \quad \text{for } n \geq 0$$

Problem: Do these sequences have limits? If so, what can you say about their limits?

(4) Compute the value of the infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{2n+1} n(2n-1)}$. Justify the steps in your calculation.

(5) Let $f : [a, b] \to \mathbb{R}$ be a C^2 function on an interval $[a, b]$. Suppose that f' is monotonically decreasing and satisfies $f' \geq m$ on $[a, b]$ for some positive constant m. Prove that

$$\left| \int_a^b \cos\,(f(x))\; dx \right| \leq \frac{4}{m}.$$

Note: To simplify the problem a bit we've made the hypotheses stronger than they need to be, and made the conclusion weaker. Thus, it is possible that you may be able to prove something even better than what is required.

(6) (a) Show that the system

$$\begin{cases} x^2 + y^2 + xe^u + e^v & = & 1 \\ u^3 + v^2 + e^{xy} & = & 2 \end{cases}$$

defines functions $u(x, y)$ and $v(x, y)$ locally about the point $(x, y) = (0, 0)$ so that $(x, y, u(x, y), v(x, y))$ is a solution of the system.

(b) Compute ∇v at $(x, y) = (0, 0)$. Explain your calculation.

(7) (a) Carefully state the definition of what it means for a family $\{f_\alpha\}_{\alpha\in I}$ of functions $f_\alpha : S\to\mathbb{R}$ to be **uniformly equicontinuous on a subset S of $\mathbb{R}$**.

Note: Many texts abbreviate the phrase 'uniformly equicontinuous' to just 'equicontinuous'.

(b) Determine whether the family of functions $f_\alpha : [0,1]\to\mathbb{R}$, given by the formula

$$f_\alpha(x) = \frac{1}{1+e^{\alpha x}} \qquad \text{for } x\in[0,1],\ \alpha\in[1,\infty),$$

is uniformly equicontinuous on $[0,1]$. Justify your answer.

(8) (a) Show that the set $\mathcal{O}(n)$ of all real orthogonal $n \times n$ matrices is compact.

(b) Show that each tangent vector M to $\mathcal{O}(n)$ at the identity matrix is a skew-symmetric matrix; that is, it satisfies the condition $M^T = -M$.

(9) (a) Carefully state the 'Method of Lagrange Multipliers'. Be sure to include all the relevant hypotheses.

(b) Determine the maximum value, on the unit sphere $S^{n-1} = \{\mathbf{x}\in\mathbb{R}^n : ||\mathbf{x}||_2 = 1\}$, of the function $f : \mathbb{R}^n\to\mathbb{R}$ given by the formula $f(\mathbf{x}) = \sum_{j=1}^{n} x_j$ for each $\mathbf{x} = (x_1, \ldots x_n)\in\mathbb{R}^n$. Be sure to justify why the value you obtain is really the desired maximum.