Algebra Qualifying Exam, Fall 2007

Student Name:

Do as many problems as you can. Although some partial credits might be given, complete solutions are much preferred.

1 (10 points). Let D_8 be the dihedral group of 8 elements.

a) List all conjugacy classes in D_8.

b) List all irreducible characters of D_8.

2 (10 points). Show that every finite field is perfect, i.e., every extension of finite fields is separable.

3 (10 points). Let p be an odd prime number.

a) Show that $\mathbf{Q}(e^{2\pi i/p})$ contains a unique quadratic extension of $\mathbf{Q}$.

b) Find a field F such that $\mathrm{Gal}(F/\mathbf{Q}) = \mathbf{Z}/3\mathbf{Z}$. Prove your answer.

4 (10 points). Prove that no group of order 462 is simple.

5 (10 points). For each pair of rings below, either prove that they are isomorphic or prove that they are not isomorphic.

a) $\mathbf{Z}/24\mathbf{Z}$, $\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/8\mathbf{Z}$.

b) $\mathbf{Z}[\sqrt{-1}]$, $\mathbf{Z}[x]/(x^2)$.

c) $\mathbf{C}$, $\mathbf{R}[x]/(x^2 + x + \pi)$.

6 (10 points). Let V be a finite dimensional vector space of dimension n over $\mathbf{Q}$. Let $A \in \mathrm{End}_{\mathbf{Q}}(V)$ be a linear map from V to itself. Assume that $A^5 = I$ (the identity). Assume further that if $v \in V$ such that $Av = v$, then $v = 0$. Show that the dimension n is divisible by 4.

7 (10 points). Let T be a linear operator on an n-dimensional vector space V over a field F. Assume that T is nilpotent, i.e., there is some positive integer k such that $T^k = 0$. Show that $T^n = 0$. (Hint: consider the decreasing filtration of V defined by the images $T^k(V)$.)

8 (10 points). Let G be a finite group acting on a finite set S. Let $\mathbf{C}[S]$ be the vector space generated by S over $\mathbf{C}$. Let χ be the character of the corresponding representation of G on $\mathbf{C}[S]$.

a) Show that for $\sigma \in G$, the value $\chi(\sigma)$ is the number of fixed points of σ in S.

b) Show also that the inner product $< \chi, 1_G >$ is the number of G-orbits in S.

9 (10 points). Let K be the splitting field over $\mathbf{Q}$ of the polynomial $f(x) = (x^2 - 2x - 1)(x^2 - 3)$. Determine the Galois group G of $f(x)$ and determine all intermediate fields explicitly.

10 (10 points). Classify up to isomorphisms all groups of order 6.