ALGEBRA QUALIFYING EXAM

September 17, 2008

Instructions: JUSTIFY YOUR ANSWERS. LABEL YOUR ANSWERS CLEARLY. Do as many problems as you can, as completely as you can. The exam is two and one-half hours. No notes, books, or calculators.

Notation: Let $\mathbb{F}_q$ denote the finite field with q elements. Let $\mathbb{Z}$ denote the integers. Let $\mathbb{Q}$ denote the rational numbers. Let $\mathbb{R}$ denote the real numbers. Let $\mathbb{C}$ denote the complex numbers.

1. Give brief justifications for each of the following:
   (a) Give an example of two rings that not isomorphic to each other, but whose underlying additive groups are isomorphic.
   (b) Give an example of an integral domain R and an ideal I of R that is prime but not maximal.
   (c) Give an example of a group G and subgroups $H \subseteq K \subseteq G$ such that K is normal in G and H is normal in K, but H is not normal in G.

2. For each of the following groups G, compute the number of subgroups of G (including trivial subgroups).
   (a) G is a cyclic group of order 63.
   (b) $G = \mathbb{F}_{17}^3$.
   (c) $G = D_8$, the dihedral group of order 8.

3. Prove that there are no simple groups of order 30.

4. Determine the splitting field over $\mathbb{Q}$ of $x^4 - 3$. Then determine the Galois group over $\mathbb{Q}$ of $x^4 - 3$, both as an abstract group and as a set of automorphisms. Give the lattice of subgroups and the lattice of subfields. Make clear which subfield is the fixed field of which subgroup.

5. Suppose G_1 and G_2 are finite groups, and the orders of G_1 and G_2 are relatively prime.
   (a) Prove that if H is a subgroup of $G_1 \times G_2$, then there are subgroups H_1 of G_1 and H_2 of G_2 such that $H = H_1 \times H_2$.
   (b) Give an example to show that the assertion in (a) is false if we do not require that the orders of G_1 and G_2 are relatively prime.

6. Suppose that G is a finite group and suppose that $H \neq \{1\}$ is a subgroup that is contained in every subgroup $K \neq \{1\}$ of G.
   (a) Prove that the order of G is a power of some prime p, and G has exactly $p - 1$ elements of order p.
   (b) Give an example of such a G and H, where G is nonabelian of order 8.

7. Find all prime ideals in the ring $\mathbb{Z}[X]/(X^3)$.

8. Suppose $L \subset \mathbb{C}$ is a finite Galois extension of $\mathbb{Q}$ with cyclic Galois group. Suppose K is a subfield of L and $K \not\subset \mathbb{R}$. Prove that:
   (a) $[K : \mathbb{Q}]$ is even.
   (b) $[L : K]$ is odd.

9. Suppose p is an odd prime. Show that there are exactly 5 groups of order $2p^2$, up to isomorphism.

10. Let A be a square matrix with entries in $\mathbb{Q}$ and with characteristic polynomial $(x^3 + 2)^2(x^2 - 3)$.
    (a) What are the possibilities for the minimal polynomial of A?
    (b) What are the trace and determinant of A?
    (c) How many distinct conjugacy classes of such matrices are there in $\mathrm{GL}_8(\mathbb{Q})$? For each conjugacy class, give one matrix A in that conjugacy class.