Algebra Qualifying Exam, Spring 2007

Student Name:

Do as many problems as you can. Although some partial credits might be given, complete solutions are much preferred.

1 (10 points). Let $\mathbf{Q}$ be the field of rational numbers. Find a field $F$ such that $\text{Gal}(F/\mathbf{Q}) = D_8$, the dihedral group with 8 elements. Prove your answer.

2 (10 points). Let $\mathbf{F}_q$ denote the finite field of $q$ elements. Show that the order of the special linear group $SL_n(\mathbf{F}_q)$ is

$$ q^{n(n-1)/2} \prod_{i=2}^{n} (q^i - 1), $$

and the order of the projective special linear group $PSL_n(\mathbf{F}_q)$ is

$$ \frac{1}{(n,q-1)} q^{n(n-1)/2} \prod_{i=2}^{n} (q^i - 1). $$

3 (5 points). Let $p$ be an odd positive integer. Show that if $n$ is an integer such that $p$ divides $n^2 + 1$, then $p \equiv 1 \pmod{4}$.

4 (15 points). Let $M$ be an $8 \times 8$ matrix with entries in $\mathbf{Q}$, with minimal polynomial $(x^4 + 1)(x + 1)^2$.
   a) What is the characteristic polynomial of $M$?
   b) What are the trace and determinant of $M$?
   c) How many conjugacy classes are there of matrices in $\text{GL}_8(\mathbf{Q})$ with this minimal polynomial? Write down one matrix from each conjugacy class.

5 (10 points). Prove that $\mathbf{Z}[\sqrt{-2}]$ is an Euclidean domain with respect to the norm map $N(a + b\sqrt{-2}) = a^2 + 2b^2$.

6 (10 points). Prove that no group of order 105 is simple.

7 (10 points). Let $F$ be a finite field and let $K$ be a finite extension of $F$. Show that both the norm map and the trace map from $K$ to $F$ are surjective. Is the same statement true if $K$ and $F$ are number fields (finite extensions of $\mathbf{Q}$)?

8 (10 points). Let $R$ be a commutative ring with identity. Let $A$ and $B$ be $n \times n$ square matrices over $R$.
   a) Assume either $A$ or $B$ is invertible. Show that the characteristic polynomials of $AB$ and $BA$ are equal.
   b) For any $A$ and $B$, not necessarily invertible, show that the characteristic polynomials of $AB$ and $BA$ are also equal.

9 (10 points). Let $G$ be a finite cyclic $p$-group and let $\rho : G \rightarrow \text{Aut}(V)$ be a representation on a finite dimensional vector space $V$ over a field of characteristic $p$. Assume that $\rho$ is irreducible. Prove that $\rho$ is trivial, i.e., $G$ acts trivially on $V$.

10 (10 points). Let $n$ be a positive integer. Prove that the polynomial $x^{4n} + 8x + 13$ is irreducible over $\mathbf{Q}$. 