Algebra Qualifying Exam, Spring 2007

Student Name:

Do as many problems as you can. Although some partial credits might be given, complete solutions are much preferred.

1 (10 points). Let $\mathbf{Q}$ be the field of rational numbers. Find a field F such that $\mathrm{Gal}(F/\mathbf{Q}) = D_8$, the dihedral group with 8 elements. Prove your answer.

2 (10 points). Let $\mathbf{F}_q$ denote the finite field of q elements. Show that the order of the special linear group $SL_n(\mathbf{F}_q)$ is

$$q^{n(n-1)/2}\prod_{i=2}^{n}(q^i-1),$$

and the order of the projective special linear group $PSL_n(\mathbf{F}_q)$ is

$$\frac{1}{(n,q-1)}q^{n(n-1)/2}\prod_{i=2}^{n}(q^i-1).$$

3 (5 points). Let p be an odd positive integer. Show that if n is an integer such that p divides n^2+1, then $p \equiv 1(\bmod 4)$.

4 (15 points). Let M be an 8×8 matrix with entries in $\mathbf{Q}$, with minimal polynomial $(x^4+1)(x+1)^2$.

a) What is the characteristic polynomial of M?

b) What are the trace and determinant of M?

c) How many conjugacy classes are there of matrices in $\mathrm{GL}_8(\mathbf{Q})$ with this minimal polynomial? Write down one matrix from each congugacy class.

5 (10 points). Prove that $\mathbf{Z}[\sqrt{-2}]$ is an Euclidean domain with respect to the norm map $N(a+b\sqrt{-2}) = a^2+2b^2$.

6 (10 points). Prove that no group of order 105 is simple.

7 (10 points). Let F be a finite field and let K be a finite extension of F. Show that both the norm map and the trace map from K to F are surjective. Is the same statement true if K and F are number fields (finite extensions of $\mathbf{Q}$)?

8 (10 points). Let R be a commutative ring with identity. Let A and B be $n\times n$ square matrices over R.

a) Assume either A or B is invertible. Show that the charactersitic polynomials of AB and BA are equal.

b) For any A and B, not necessarily invertible, show that the charactersitic polynomials of AB and BA are also equal.

9 (10 points). Let G be a finite cyclic p-group and let $\rho : G \longrightarrow \mathrm{Aut}(V)$ be a representation on a finite dimensional vector space V over a field of charactersitic p. Assume that ρ is irreducible. Prove that ρ is trivial, i.e., G acts trivially on V.

10 (10 points). Let n be a positive integer. Prove that the polynomial $x^{4^n}+8x+13$ is irreducible over $\mathbf{Q}$.