Name: 

Algebra Qualifying Exam, Spring 2008

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ALGEBRA QUALIFYING EXAM  
June 11, 2008

Instructions: JUSTIFY YOUR ANSWERS. LABEL YOUR ANSWERS CLEARLY. Do as many problems as you can, as completely as you can. The exam is two and one-half hours. No notes, books, or calculators.

Notation: Let $\mathbb{F}_q$ denote the finite field with $q$ elements. Let $\mathbb{Z}$ denote the integers. Let $\mathbb{Q}$ denote the rational numbers. Let $\mathbb{R}$ denote the real numbers.

1. Compute the following.
   
   (a) Suppose $G$ is a cyclic group of order 20. How many automorphisms does $G$ have?
   
   (b) How many homomorphisms are there from $\mathbb{Z}$ to the symmetric group $S_n$ on $n$ letters?
   
   (c) If $G$ is a group, and $g \in G$ is an element of order 25, what is the order of $g^{10}$?

2. Show that if $G$ is a group of order $2pq$, where $p$ and $q$ are (not necessarily distinct) odd primes, then $G$ is not simple.

3. Factor the polynomial $x^4 + 1 \in \mathbb{F}[x]$ and find the splitting field over $\mathbb{F}$ if the ground field $\mathbb{F}$ is:
   
   (a) $\mathbb{Q}$
   
   (b) $\mathbb{F}_2$
   
   (c) $\mathbb{R}$

4. Let $R = \mathbb{Z}[X, Y]$, the ring of polynomials over $\mathbb{Z}$ in the variables $X, Y$. For each of the following ideals, determine whether the ideal is prime and whether it is maximal. In each case give a short justification.
   
   (a) $(X, Y)$
   
   (b) $(3X, Y)$
   
   (c) $(X^2 + 1, Y)$
   
   (d) $(5, X^2 + 1, Y)$

5. Let $K$ be the splitting field of $X^{40} - 1$ over $\mathbb{Q}$. Determine the number of fields $F$ such that $\mathbb{Q} \subseteq F \subseteq K$.

6. Suppose $G$ is a finite group and $H \neq G$ is a subgroup containing every subgroup $K \neq G$ of $G$.
   
   (a) Prove that the order of $G$ is a prime power.
   
   (b) Prove that if $G$ is abelian then $G$ is cyclic.

7. Find all prime ideals in the ring $\mathbb{Z} \times \mathbb{Z}$.

8. Let $S$ be the set of all $6 \times 6$ matrices $A$ with entries in $\mathbb{Q}$ such that the characteristic polynomial of $A$ is $x^6 - x^2$ and the minimal polynomial of $A$ is $x^3 - x$.
   
   (a) If $A, B \in S$, show that $A$ and $B$ are similar.
   
   (b) Give an example of an element of $S$.
   
   (c) If $A \in S$, what is the dimension of the null space of $(A^2 + 1)^2$?

9. Show that the quaternion group $Q_8$ of order 8 is not a semidirect product of two proper subgroups.

10. Suppose $F$ is an algebraically closed field. Find all monic separable polynomials $f(x) \in F[x]$ such that the set of zeros of $f(x)$ is closed under multiplication.