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Qualifying Examination/ Complex Analysis

September 18, 2006

Table of your scores

Problem 1 ________/ 10

Problem 2 ________/ 10

Problem 3 ________/ 10

Problem 4 ________/ 10

Problem 5 ________/ 10

Problem 6 ________/ 10

Problem 7 ________/ 10

Problem 8 ________/ 10

Total ________/ 80

Notation:

$D(z_0, r) = \{z \in \mathbf{C} : |z - z_0| < r\}$ and $\overline{D}(z_0, r) = \{z \in \mathbf{C} : |z - z_0| \leq r\}$.

1. Show that $\sum\limits_{n=1}^{\infty} \frac{1}{z^2+n^2}$ is meromorphic function on $\mathbf{C}$.

2. Show that for $a > 0$,

$$\int_0^\infty \frac{\cos ax}{(1+x^2)^2}\,dx = \frac{\pi(1+a)}{4e^a}.$$

3. Let $P(z)$ be a polynomial in z. Assume that $P(z) \neq 0$ for $\text{Re}(z) > 0$. Show that $P'(z) \neq 0$ for $\text{Re}(z) > 0$.

4. Let $z_1, \ldots, z_n$ be distinct complex numbers contained in the disk $D(0, R)$. Let f be analytic in the closed disk $\overline{D}(0, R)$. Let

$$Q(z) = (z - z_1) \ldots (z - z_n).$$

Prove that

$$P(z) = \frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=R} f(\zeta) \frac{1 - \frac{Q(z)}{Q(\zeta)}}{\zeta - z} \, \mathrm{d}\zeta$$

is a polynomial of degree $n - 1$ having the same values as f at the points $z_1, \ldots, z_n$.

5. Let f be a function analytic in the unit disc $D(0,1)$ and $|f(z) - z| \leq 1$ on the unit circle $\partial D(0,1)$. Show that $\left|f'\left(\frac{1}{2}\right)\right| \leq 7/3$.

6. Let real $a > 1$. Prove that the equation $ze^{a-z} = 1$ has a single solution in the closed unit disk $\overline{D}(0,1)$ which is real and positive.

7. Let Ω be a bounded domain in **C**, and let $\{f_j\}_{j=1}^{\infty}$ is a sequence of analytic functions on Ω such that

$$\int_{\Omega} |f_j(z)|^2 dA(z) \leq 1.$$

Prove that $\{f_j\}_{j=1}^{\infty}$ is a normal family in Ω.

8. Let $f : D(0,1) \to \mathbf{C}$ be a bounded analytic function. Let a_n be the non-zero zeros of f in D counted according to multiplicity. Prove

$$\sum_{n=1}^{\infty}(1 - |a_n|) < \infty$$