Complex Analysis Qualifying Examination

Choose any 8 problems from 9

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1. Prove the following Jordan's lemma. Let $f(z)$ be continuous in the region $D = \{z \in \mathbb{C} : |z| \geq R_0, \text{Im} z \geq 0\}$ and $\lim_{z \to \infty} f(z) = 0$ uniformly on $D$. Then for any positive number $a$

$$\lim_{R \to \infty} \int_{\Gamma_R} e^{iaz} f(z) \, dz = 0,$$

where $\Gamma_R$ is the arc of the circle $\{z \in \mathbb{C} : |z| = R\}$, which lies in the semi-plane $\text{Im} z \geq 0$. 
2. Let $f(z)$ be holomorphic in the closed unit disc $\overline{D(0,1)}$. Prove

$$f(z) = \frac{1}{\pi} \int_{D(0,1)} \frac{f(w)}{(1 - z\overline{w})^2} dA(w), \quad z \in D(0,1).$$
3. Let $\alpha, \beta$ and $\gamma$ are positive real numbers. Then find the radius of convergence for the series
\[ \sum_{n=0}^{\infty} \frac{\alpha (\alpha + 1) \cdots (\alpha + n - 1) \beta (\beta + 1) \cdots (\beta + n - 1)}{n! \gamma (\gamma + 1) \cdots (\gamma + n - 1)} z^n. \]
4. Show that

\[ F(z) = \int_0^1 \frac{e^{tz}}{1 + t} dt \]

is holomorphic in \( \mathbb{C} \).
5. Let $f : D(0, 1) \to D(0, 1)$ be holomorphic. Prove

$$\left| \frac{f(z) - f(w)}{1 - f(z)f(w)} \right| \leq \left| \frac{z - w}{1 - zw} \right|, \quad z, w \in D(0, 1).$$
6. Let $\Omega \neq \mathbb{C}$ be a simply connected domain in $\mathbb{C}$. Let $f : \Omega \rightarrow \Omega$ be a holomorphic mapping which fixes two distinct points in $\Omega$ (i.e. there are $p, q \in \Omega$ so that $f(p) = p$ and $f(q) = q$). Show that $f(z) \equiv z$ on $\Omega$. 
7. Let $a$ be a real number, evaluate the following integral

$$\int_{0}^{\infty} \frac{\sin ax}{\sinh x} \, dx.$$
8. Let $f(z)$ be analytic on $C\{1\}$ and have a simple pole at $z = 1$ with residue $\lambda$. Prove that for every $R > 0$,

$$\lim_{n \to +\infty} R^n \left| (-1)^n \frac{f^{(n)}(2)}{n!} - \lambda \right| = 0.$$
9. Suppose that $f(z)$ is an entire function such that

$$|f(z)| \leq Be^{A|z|}, \quad z \in \mathbb{C}$$

for some positive numbers $A, B$. Let $\omega_1, \omega_2, \cdots$ be the zeros of $f$ listed with appropriate multiplicity. Prove that

$$\sum_{n=1}^{\infty} (1 + |\omega_n|)^{-\alpha} < \infty$$

for all $\alpha > 1$. 