Complex Analysis

Qualifying Exam

Thursday, September 18, 2008 — 10:00 am - 12:30 pm, Rowland Hall 306

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Student’s name:
Problem 1.

Compute the area of the image of the unit disc $D = \{z \mid |z| < 1\}$ under the map $f(z) = z + \frac{z^2}{2}$. 
Problem 2.

Find all entire functions $f(z)$ that satisfy

\[ f''\left(\frac{1}{n}\right) = 4f\left(\frac{1}{n}\right) \]

for all $n \in \mathbb{N}$. 
Problem 3.

Let \( L \subset \mathbb{C} \) be the line \( L = \{ z = x + iy \mid x = y \} \). Assume that \( f : \mathbb{C} \to \mathbb{C} \) is an entire function such that for any \( z \in L \) we have \( f(z) \in L \). Assume that \( f(1) = 0 \). Prove that \( f(i) = 0 \).
Problem 4.

Find the largest disk centered at 1 in which the Taylor series for

\[ \frac{1}{1 + z^2} = \sum a_k (z - 1)^k \]

will converge. (Hint: you do not actually have to find the coefficients \(a_k\) nor the full series to answer this question.)
Problem 5.

Evaluate the integral

\[
\int_{0}^{+\infty} \left(\frac{\sin x}{x}\right)^2 dx
\]
Problem 6.

Suppose a function \( f : D \rightarrow D \), where \( D = \{ |z| < 1 \} \) is the unit disc, is holomorphic and \( f(0) = \alpha \neq 0 \). Show that \( f \) cannot have a zero in the open disk \( D(0, |\alpha|) = \{ |z| < |\alpha| \} \).
Problem 7.

Let $u$ be a harmonic function on $\mathbb{R}^2$ that does not take zero value (i.e. $u(x) \neq 0 \ \forall x \in \mathbb{R}^2$). Show that $u$ is constant.
Problem 8.

How many zeros does the function \( f(z) = 14z^{100} - 5e^z \) have in the unit disc? What are the multiplicities of zeros?