COMPLEX ANALYSIS

Qualifying Exam

Thursday, September 18, 2008 — 10:00 am - 12:30 pm, Rowland Hall 306

Problem	1	2	3	4	5	6	7	8	Σ
Points									

Student's name:

Problem 1.

Compute the area of the image of the unit disc $D = \{z \mid |z| < 1\}$ under the map $f(z) = z + \frac{z^2}{2}$.

Problem 2.

Find all entire functions $f(z)$ that satisfy

$$f''\left(\frac{1}{n}\right) = 4f\left(\frac{1}{n}\right) \quad \text{for all} \quad n \in \mathbb{N}.$$

Problem 3.

Let $L \subset \mathbb{C}$ be the line $L = \{z = x + iy \mid x = y\}$. Assume that $f : \mathbb{C} \to \mathbb{C}$ is an entire function such that for any $z \in L$ we have $f(z) \in L$. Assume that $f(1) = 0$. Prove that $f(i) = 0$.

Problem 4.

Find the largest disk centered at 1 in which the Taylor series for

$$\frac{1}{1+z^2} = \sum a_k (z-1)^k$$

will converge. (Hint: you do not actually have to find the coefficients a_k nor the full series to answer this question.)

Problem 5.

Evaluate the integral

$$\int_0^{+\infty} \left(\frac{\sin x}{x} \right)^2 dx$$

Problem 6.

Suppose a function $f : D \to D$, where $D = \{|z| < 1\}$ is the unit disc, is holomorphic and $f(0) = \alpha \neq 0$. Show that f cannot have a zero in the open disk $D(0, |\alpha|) = \{|z| < |\alpha|)$.

Problem 7.

Let u be a harmonic function on $\mathbb{R}^2$ that does not take zero value (i.e. $u(x) \neq 0 \ \forall x \in \mathbb{R}^2$). Show that u is constant.

Problem 8.

How many zeros does the function $f(z) = 14z^{100} - 5e^z$ have in the unit disc? What are the multiplicities of zeros?