Print Your Name: _______________ ________________
last first

Print Your I.D. Number: __________ _______________

Qualifying Examination/ Complex Analysis

September, 2009

Table of your scores

Problem 1 ————/ 10

Problem 2 ————/ 10

Problem 3 ————/ 10

Problem 4 ————/ 10

Problem 5 ————/ 10

Problem 6 ————/ 10

Problem 7 ————/ 10

Problem 8 ————/ 10

Total ————/ 80

Notation: Let $D(z_0, r)$ be the disk centered at z_0 and radius r in the complex plane $\mathbf{C}$.

1. For $\alpha, \beta, \gamma > 0$, find the radius of convergence for the series

$$\sum_{n=0}^{\infty} \frac{\alpha\,(\alpha+1)\ldots(\alpha+n-1)\,\beta\,(\beta+1)\ldots(\beta+n-1)}{n!\gamma\,(\gamma+1)\ldots(\gamma+n-1)} z^n.$$

2. Prove or disprove there is a holomorphic function $f(z)$ on the unit disc $D(0,1)$ so that

$$\left\{z \in D(0,1) : f^{(k)}(z) = 0 \text{ for some non-negative integer } k\right\} = (-1,1)$$

where $f^{(k)}$ is the kth derivative of f.

3. Let $L \subset \mathbf{C}$ be the line $L = \{z = x + iy \mid y = 2\}$. Assume that $f : \mathbf{C} \to \mathbf{C}$ is an entire function such that for any $z \in L$ we have $f(z) \in L$. Assume that $f(0) = i$. Find $f(4i)$.

4. Let $\{f_\alpha\}_{\alpha\in A}$ be a family of holomorphic functions on the unit disc $D(0,1)$ such that

$$\text{for all } z \in D(0,1) \;\; \forall f \in \{f_\alpha\}_{\alpha\in A} \quad \mathrm{Im} f(z) \neq (\mathrm{Re} f(z))^2 \,.$$

Prove that $\{f_\alpha\}_{\alpha\in A}$ is a normal family (i.e. every sequence in $\{f_\alpha\}_{\alpha\in A}$ has a subsequence that converges or tends to infinity uniformly on compact subsets of $D(0,1)$).

5. Let $f(z) : \mathbf{C} \setminus \{0, 1\} \to \mathbf{C} \setminus D(0, 1)$ be holomorphic. Prove that f must be a constant.

6. Suppose that f is a polynomial such that all of its zeros are inside of the unit disc. Prove that all zeros of f' are also inside of the unit disc.

7. Find the integral

$$\int_0^\infty \frac{1 - \cos(2x)}{x^2} dx$$

8. Does there exist a sequence of holomorphic functions $\{f_n(z)\}_{n=1}^{\infty}$ on the unit disc $D(0,1)$ so that $f_n(z) \to 1/z$ uniformly on $\{z \in \mathbf{C} : |z| = 1/2\}$ as $n \to \infty$?