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Qualifying Examination/ Complex Analysis

June 20, 2006

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Notation: Let $D(z_0, r)$ be an open disk in the complex plane centered at z_0 with radius r.

1. Prove or disprove that there exists an analytic function $f(z)$ in the unit disk $D(0,1)$ such that

$$f\Big(\frac{1}{n}\Big) = f\Big(-\frac{1}{n}\Big) = \frac{1}{n^3}, \quad \text{for all} \quad n = 1, 2, 3, \cdots.$$

2. Complete the following problems:
   (a) State the Liouville's theorem
   (b) Prove the Liouville's theorem by calculating the following integral

$$\int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz$$

and taking the limit $R \to \infty$.

3. The Bernoulli polynomials $\phi_n(z)$ are defined by the expansion

$$\frac{e^{tz}-1}{e^t-1} = \sum_{n=1}^{\infty} \frac{\phi_n(z)}{n!} t^{n-1}.$$

Prove the following two statements

(i) $\phi_n(z+1) - \phi_n(z) = nz^{n-1}$;

(ii) $\frac{\phi_{n+1}(n+1)}{n+1} = 1 + 2^n + 3^n + \cdots n^n$.

4. Let $f(z)$ be analytic and satisfy $|f(z)| \leq 100|z|^{-2}$ in the strip $\alpha_1 \leq \operatorname{Re} z \leq \alpha_2$. Prove the function

$$h(x) = \int_{-\infty}^{\infty} f(x + i\,y)dy$$

is a constant function of $x \in [\alpha_1, \alpha_2]$.

5. Evaluate the integral:

$$\int_0^\infty \frac{\sin^2 x}{x^2}\, dx.$$

6. Prove or disprove that there is a sequence of analytic polynomial $\{p_n(z)\}_{n=1}^{\infty}$ so that $p_n(z) \to \overline{z}^4$ as $n \to \infty$ uniformly for $z \in \partial D(0,1) = \{z \in \mathbf{C} : |z| = 1\}$.

7. Let f be analytic in the unit disc $D(0,1)$ and continous on $\overline{D(0,1)}$. Assume that

$$|f(z)| = |e^z| \quad \text{for all } z \in \partial D(0,1) = \{z \in \mathbf{C} : |z| = 1\}$$

Find all such f.

8. Let $f(z)$ be an entire analytic function and satisfy

$$f(z+1) = f(z) \quad \text{and} \quad |f(z)| \leq e^{|z|}, \quad z \in \mathbf{C}.$$

Prove that $f(z)$ must be constant. Here $\mathbf{C}$ denotes the the whole complex plane.