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Qualifying Examination/ Complex Analysis

June, 2009

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Notation: Let $D(z_0, r)$ be the disk centered at z_0 and radius r in the complex plane $\mathbf{C}$, $\mathbf{N}$ is the set of all positive integers, let $U = \{z \in \mathbf{C} : \text{Im z} > 0\}$ be the upper half plane.

1. (a) State the Rouche's Theorem.
   (b) Let $a > e$ be a real number. Prove that the equation

$$a\, z^4\, e^{-z} = 1$$

has a single solution in $D(0, 1)$, which is real and positive.

2. Suppose that a function $f(z)$ is holomorphic in the unit disc $D(0,1)$ and has the property

$$f\left(\frac{1}{2n}\right) = f^{(4)}\left(\frac{1}{2n}\right) \quad \text{for all} \quad n \in \mathbf{N}.$$

Prove that f can be extended to an entire function on $\mathbf{C}$. (Here $f^{(4)} = \frac{\partial^4 f}{\partial z^4}$.)

3. If $f(z)$ is continuos in the region $\mathrm{Re}z \geq \sigma$ (σ is a fixed real number) and $\lim_{z\to\infty} f(z) = 0$, then for any negative number t

$$\lim_{R\to\infty} \int\limits_{\Gamma_R} e^{tz} f(z)\, dz = 0,$$

where Γ_R is the arc of the circle $|z| = R$, $\mathrm{Re}z \geq \sigma$.

4. (a) State the Riemann mapping theorem.

(b) Find explicitly a conformal mapping of the domain

$$\{z \in \mathbf{C} \mid |z| < 1, \quad \operatorname{Re} z > 0,\ \operatorname{Im} z > 0\}$$

to the unit disc.

5. Does there exist a conformal automorphism φ of the unit disc such that $\varphi(1/2) = 0$ and $\varphi(0) = \frac{i}{3}$?

1. Find the integral $\int_0^\infty \frac{x \cos ax}{\sinh x} dx$.

3. Prove that if $|a| \neq R$, then

$${}_{|z|=R} \frac{|dz|}{|z-a|\,|z+a|} \leq \frac{2\pi R}{\left|R^2 - |a|^2\right|}.$$

3. Let $0 < a < 1$ be any real number. Then

(a) Prove the following identity:

$$\int_0^{2\pi} \frac{1}{1 + a^2 - 2a\cos\theta} d\theta = \frac{2\pi}{1 - a^2}$$

(b) Find the limit

$$\lim_{k \to +\infty} \int_{|z|=(k+\frac{1}{2})\pi} \frac{\pi}{z^2 \sin z} dz$$

1. Let $f(z)$ be a non-constant entire function on $\mathbf{C}$. Use Liouville's Theorem to prove that the image of f (or $f(\mathbf{C})$) is dense in $\mathbf{C}$.
4. (a) State the Schwarz reflection principle for holomorphic function on the unit disk.

(b) Let $f(z)$ be holomorphic in the unit disc $D(0,1)$ and continuous on the closed disc $\overline{D(0,1)}$. Prove or disprove there exists such f so that $f(e^{i\theta}) = e^{-i\theta}$ for $0 < \theta < \pi/4$.

5. Let D be a bounded domain in $\mathbf{C}$ with $0 \in D$. If $f : D \to D$ is a holomorphic map so that $f(0) = 0$ and $f'(0) = 1$. Show that $f(z) = z$ on D.

6. Let $f(z)$ be holomorphic on a domain D in the complex plane. If $|f(z)|^2$ is harmonic in D. What can you conclude on f? (show your work.)

7. Let f be an entire function on $\mathbf{C}$ with $|f(z)| = 1$ for $|z| = 1$ and $f'''(0) = 6$ (the third order derivative of f at $z = 0$). Find all such f.