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Real Analysis Qualifying Exam
September 19, 2008

Problem #	Points
1	
2	
3	
4	
5	
6	
Total	

Instructions. *Do all problems if possible. Use only one side of each sheet. Do at most one problem on each page. Write your name on every page. Justify your answers. Where appropriate, state without proof results that you use in your solutions.*

1. Let f be a Lebesgue integrable function of the real line. Prove that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\, \sin(nx)\, \mathrm{d}x = 0.$$

2. Let g be an absolutely continuous monotone function on $[0,1]$. Prove that, if $E \subset [0,1]$ is a set of Lebesgue measure zero, then the set $g(E) = \{g(x);\, x \in E\} \subset \mathbb{R}$ is also a set of Lebesgue measure zero.

3. Let ν be a finite Borel measure on the real line, and set $F(x) = \nu\{(-\infty, x]\}$. Prove that ν is absolutely continuous with respect to the Lebesgue measure μ_L if and only F is an absolutely continuous function. In this case show that its Radon-Nikodym derivative is the derivative of F, that is, $\frac{\mathrm{d}\nu}{\mathrm{d}\mu_L} = F'$ almost everywhere.

4. Let μ be a measure and let λ, λ_1, λ_2 be signed measures on the measurable space $(X, \mathcal{A})$. Prove:

   (a) If $\lambda \perp \mu$ and $\lambda \ll \mu$ then $\lambda = 0$.

   (b) If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then, if we set $\lambda = c_1\lambda_1 + c_2\lambda_2$ with c_1, c_2 real numbers such that λ is a signed measure, we have $\lambda \perp \mu$.

   (b) If $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, then, if we set $\lambda = c_1\lambda_1 + c_2\lambda_2$ with c_1, c_2 real numbers such that λ is a signed measure, we have $\lambda \ll \mu$.

5. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $\{f_n\}_{n\in\mathbb{N}}$ and f be extended real-valued $\mathcal{A}$-measurable functions on a set $D \in \mathcal{A}$ such that $\lim_{n\to\infty} f_n = f$ on D. Then for every $\alpha \in \mathbb{R}$ we have
   (1) $\quad \mu\{D : f > \alpha\} \leq \liminf_{n\to\infty} \mu\{D : f_n \geq \alpha\}$
   (2) $\quad \mu\{D : f < \alpha\} \leq \liminf_{n\to\infty} \mu\{D : f_n \leq \alpha\}$.

6. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $\{f_n\}_{n\in\mathbb{N}}$ and f be a sequence of extended real-valued $\mathcal{A}$-measurable functions on a set $D \in \mathcal{A}$ with $\mu(D) < \infty$. Show that f_n converges to 0 in measure on D if and only if $\lim_{n\to\infty} \int_D \frac{|f_n|}{1+|f_n|} d\mu = 0$.