ALGEBRA QUALIFYING EXAM

June 19, 2006

(1a) (4 points) Give an example of an infinite group in which every element has finite order.

(1b) (4 points) Prove that the polynomial $f(x) = 1 + \frac{x}{1} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} \in \mathbb{Q}[x]$ has no multiple roots in $\mathbb{C}$.

(1c) (2 points) State Lagrange's Theorem:

(2) (10 points) Let L be the splitting field of $x^4 - 2$ over $\mathbb{Q}$.

(a) Find $[L : \mathbb{Q}]$.

(b) Describe the Galois group $\text{Gal}(L/\mathbb{Q})$, both as an abstract group and as a set of automorphisms.

(3) (10 points) For which primes p can one find a nonzero homomorphism $\mathbb{Z}[i] \to \mathbb{Z}/p\mathbb{Z}$? (Here, i denotes $\sqrt{-1}$.)

(4) (10 points) (a) Prove that every group of order 185 is abelian.

(b) How many groups of order 185 are there, up to isomorphism?

(5) (10 points) Let A, B, N be submodules of a module M and suppose that $N \subset A \cap B$. Prove that there exist natural homomorphisms $\phi : A/N \to M/B$ and $\psi : B/N \to M/A$ such that $\text{Ker}(\phi) \simeq \text{Ker}(\psi)$.

(6) (10 points) Determine the structure (as a direct product of cyclic groups) of the group of units of the ring $\mathbb{F}_5[x]/(x^3 - 1)\mathbb{F}_5[x]$.

(7) (10 points) Suppose that A is a 3×3 matrix with entries in $\mathbb{C}$. Suppose further that A is not diagonalizable, $\text{trace}(A) = 3$, and $\det(A) = 1$.

(a) List all possibilities for the characteristic polynomial of A.

(b) List all possibilities for the minimal polynomial of A.

(c) List all possibilities for the Jordan canonical form of A.

(8) (10 points) Let q be a prime power and n a positive integer.

(a) Prove that the map ϕ defined by $\phi(x) = x^q$ is an automorphism of $\mathbb{F}_{q^n}$ that fixes $\mathbb{F}_q$.

(b) Prove that the automorphism ϕ of part (a) generates $\text{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$.

(9) (20 points) For each of the following 4 statements, answer TRUE or FALSE and justify your answer with a proof or counterexample.

(a) Every Euclidean domain is a principal ideal domain.

(b) For every commutative ring R with identity, every subring of R is an ideal of R.

(c) For every commutative ring R with identity, every maximal ideal of R is a prime ideal of R.

(d) If G is a group, H is a normal subgroup of G, and K is a normal subgroup of H, then K is a normal subgroup of G.