Real Analysis Qualifying Examination
Spring 2006
June 21, 2006, 10 am -12.30 pm

Instructions: Do all problems. Use only one side of each page. Write your name on each page. Do at most one problem on each page. Justify your answers. Where appropriate, state without proof, results that you use in your solutions.

1. Let $f \in L^p \cap L^q$ with $1 \le p < q \le \infty$. Prove that $f \in L^r$ for all $p < r < q$.

2. Let $\mathcal{M}$ denote the Lebesgue measurable sets on the real line. Consider two measures on $\mathcal{M}$: Lebesgue measure m and the counting measure τ, where for $A \in \mathcal{M}$ we set $\tau(A)$ to be the number of points in A. Show that m is absolutely continuous with respect to τ, but that $\frac{dm}{d\tau}$ does not exist, i.e., there is no measurable function h such that $m(A) = \int_A h\, d\tau$ for all $A \in \mathcal{M}$. Does this contradict the Radon-Nikodym theorem?

3. Given a measure space $(X, \mathfrak{A}, \mu)$, let $(f_n : n \in \mathbb{N})$ and f be extended real-valued $\mathfrak{A}$-measurable functions on a set $D \in \mathfrak{A}$ and assume that f is real-valued a.e. on D. Suppose there exists a sequence of positive numbers $(\varepsilon_n : n \in \mathbb{N})$ such that
   1° $\sum_{n\in\mathbb{N}} \varepsilon_n < \infty$,
   2° $\int_D |f_n - f|^p \, d\mu < \varepsilon_n$ for every $n \in \mathbb{N}$ for some fixed $p \in (0, \infty)$.
   Show that the sequence $(f_n : n \in \mathbb{N})$ converges to f a.e. on D.

4. Let f be a real-valued continuous function of bounded variation on $[a, b]$. Suppose f is absolutely continuous on $[a + \eta, b]$ for every $\eta \in (0, b - a)$. Show that f is absolutely continuous on $[a, b]$.

5. Let $\mathcal{A}$ be a collection of pairwise disjoint subsets of a σ-finite measure space, each of positive measure. Show that $\mathcal{A}$ is at most countable.

6. Let $(X, \mathcal{M}, \mu)$ be a complete measure space and let f be a nonnegative integrable function on X. Let $b(t) = \mu\{x \in X \ : \ f(x) \ge t\}$. Show that

$$\int f \, d\mu = \int_0^\infty b(t)\, dt$$