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Real Analysis Qualifying Exam
June 16, 2010

Problem #	Points
1	
2	
3	
4	
5	
6	
Total	

Instructions. *Do all problems if possible. Use only one side of each sheet. Do at most one problem on each page. Write your name on every page. Justify your answers. Where appropriate, state without proof results that you use in your solutions.*

Prob. 1. Suppose f is Lipschitz continuous in $[0, 1]$, that is, $|f(x) - f(y)| \le L|x - y|$ for some constant L. Show that
(i) $m\big(f(E)\big) = 0$ if $m(E) = 0$.
(ii) If E is measurable, then $f(E)$ is also measurable.

Prob. 2. Let $\{q_k\}$ be all the rational numbers in $[0, 1]$. Show that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \frac{1}{\sqrt{|x - q_k|}} \text{ converges a.e. in } [0, 1].$$

Prob. 3. Suppose that $f(t) = \int_0^t g(s)\, ds$ where $g(s)$ is integrable over $[0, 1]$. Show that

$$\lim_{n \to \infty} \sum_{k=0}^{2^n - 1} \left| f\Big(\frac{k+1}{2^n}\Big) - f\Big(\frac{k}{2^n}\Big) \right|^2 = 0.$$

Prob. 4. Let f be a real-valued uniformly continuous function on $[0, \infty)$. Show that if f is Lebesgue integrable on $[0, \infty)$, then $\lim_{x \to \infty} f(x) = 0$.

Prob. 5. Let $(X, \mathfrak{A}, \mu)$ be a measure space and let f be an extended real-valued $\mathfrak{A}$-measurable function on X such that $\int_X |f|^p \, d\mu < \infty$ for some $p \in (0, \infty)$. Show that

$$\lim_{\lambda \to \infty} \lambda^p \mu\{X : |f| \ge \lambda\} = 0.$$

Prob. 6. Consider the Lebesgue measure space $(\mathbb{R}, \mathfrak{M}_L, \mu_L)$ on $\mathbb{R}$. Let f be a μ_L-integrable extended real-valued $\mathfrak{M}_L$-measurable function on $\mathbb{R}$. Show that

$$\lim_{h \to 0} \int_{\mathbb{R}} |f(x+h) - f(x)| \mu_L(dx) = 0.$$