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Real Analysis Qualifying Exam

Sep. 21, 2007

Problem #	Points
1	
2	
3	
4	
5	
6	
Total	

Instructions. *Do all problems if possible. Use only one side of each sheet. Do at most one problem on each page. Write your name on every page. Justify your answers. Where appropriate, state without proof results that you use in your solutions.*

Prob. 1. Let $1 \le p < q < \infty$. Which of the following statements are true and which are false? Justify.
1. $L^p\big(\mathbb{R}, \mathfrak{M}_L, \mu_L\big) \subset L^q\big(\mathbb{R}, \mathfrak{M}_L, \mu_L\big)$.
2. $L^q\big(\mathbb{R}, \mathfrak{M}_L, \mu_L\big) \subset L^p\big(\mathbb{R}, \mathfrak{M}_L, \mu_L\big)$.
3. $L^p\big([2,5], \mathfrak{M}_L, \mu_L\big) \subset L^q\big([2,5], \mathfrak{M}_L, \mu_L\big)$.
4. $L^q\big([2,5], \mathfrak{M}_L, \mu_L\big) \subset L^p\big([2,5], \mathfrak{M}_L, \mu_L\big)$.

Prob. 2. Does there exist a Lebesgue measurable subset A of $\mathbb{R}$ such that for every interval (a,b) we have $\mu_L\big(A \cap (a,b)\big) = \frac{b-a}{2}$? Either construct such a set or prove it does not exist.

Prob. 3. Let f be a real valued measurable function on the finite measure space (X, Σ, μ). Prove that the function $F(x,y) = f(x) - 5f(y) + 4$ is measurable in the product measure space $(X \times X, \sigma(\Sigma \times \Sigma), \mu \times \mu)$, and that F is integrable if and only if f is integrable.

Prob. 4. Let $f \in \mathrm{L}^{\frac{3}{2}}\big([0,5], \mathfrak{M}_L, \mu_L\big)$. Prove that

$$\lim_{t \downarrow 0} \frac{1}{t^{\frac{1}{3}}} \int_0^t f(s)\,\mathrm{d}s = 0.$$

Prob. 5. Let $(f_n : n \in \mathbb{N})$ be a sequence of real-valued functions and f be a real-valued function on $[a,b]$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for $x \in [a,b]$. Let $V_a^b(f)$ be the total variation of f on $[a,b]$. Show that

$$V_a^b(f) \le \liminf_{n\to\infty} V_a^b(f_n).$$

Prob. 6. Let $f \in L^1(\mathbb{R}, \mathfrak{M}_L, \mu_L)$. With $h > 0$ fixed, define a function φ_h on $\mathbb{R}$ by setting

$$\varphi_h(x) = \frac{1}{2h} \int_{[x-h,x+h]} f(t)\,\mu_L(dt) \quad \text{for } x \in \mathbb{R}$$

(a) Show that φ_h is $\mathfrak{M}_L$-measurable on $\mathbb{R}$.
(b) Show that $\varphi_h \in L^1(\mathbb{R}, \mathfrak{M}_L, \mu_L)$ and $\|\varphi_h\|_1 \le \|f\|_1$.