Algebra Qualifying Exam

September 22, 2004

There are 11 questions, each worth 12 points (divided equally if the problem has multiple parts). You are not expected to answer all the questions; do as many as you can, as completely as you can. You have two and one-half hours. Good luck.

Notation: $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ denote the rings of integers, rational numbers, real numbers, and complex numbers, respectively. If R is a ring and n is a positive integer, then $\mathrm{GL}_n(R)$ and $\mathrm{SL}_n(R)$ denote the groups of invertible $n \times n$ matrices, and $n \times n$ matrices with determinant 1, respectively, with entries in R.

1. For each of the following rings, list all maximal ideals.
   (a) $\mathbf{Z}/90\mathbf{Z}$
   (b) $\mathbf{Q}[x]/(x^2+1)$
   (c) $\mathbf{C}[x]/(x^2+1)$
   (d) $\mathbf{Q}[x]/(x^3+x^2)$

2. Let M be a 9×9 matrix over $\mathbf{C}$ with characteristic polynomial $(x^2+1)^3(x+1)^3$ and minimal polynomial $(x^2+1)^2(x+1)$.
   (a) Find $\mathrm{trace}(M)$ and $\det(M)$.
   (b) How many distinct conjugacy classes of such matrices are there in $\mathrm{GL}_9(\mathbf{C})$? Explain.
   (c) Write down a 9×9 matrix with coefficients in $\mathbf{Q}$ having the above characteristic and minimal polynomials.

3. Let G be a finite group of order $n > 2$. Let H be a subgroup of G such that $r = [G : H] > 1$. Assume that $r! < 2n$. Prove that G is not a simple group. [Hint: construct a map from G into the symmetric group S_r.]

4. Let F be the splitting field of $x^{10}-1$ over $\mathbf{Q}$. Find $\mathrm{Gal}(F/\mathbf{Q})$, both as an abstract group, and as a group of explicitly described automorphisms of F.

5. Let $a_1, ..., a_n \in \mathbf{Z}$ ($n > 1$) such that $\gcd(a_1, ..., a_n) = 1$. Show that there is an element $A \in \mathrm{SL}_n(\mathbf{Z})$ such that the first row of A is $(a_1, ..., a_n)$.

6. (a) Prove that the three *additive groups* $\mathbf{Z} \times \mathbf{Z}$, $\mathbf{Z}[i]$, and $\mathbf{Z}[x]/(x^2)$ are all isomorphic to each other.

   (b) Prove that *no two* of the *rings* $\mathbf{Z} \times \mathbf{Z}$, $\mathbf{Z}[i]$, and $\mathbf{Z}[x]/(x^2)$ are isomorphic to each other.

7. Let $\mathbf{F}_q$ be a finite field with q elements, and K a finite extension of $\mathbf{F}_q$. Let $n = [K : \mathbf{F}_q]$.

   (a) How many elements does K have? Explain.

   (b) Show that every extension of $\mathbf{F}_q$ is separable.

   (c) Show that K is a Galois extension of $\mathbf{F}_q$.

   (d) Exhibit an automorphism σ of K of order n, such that σ restricts to the identity automorphism of $\mathbf{F}_q$. Conclude that $\mathrm{Gal}(K/\mathbf{F}_q)$ is cyclic.

8. Suppose $f(x) \in \mathbf{Q}[x]$ is irreducible and let K denote its splitting field.

   (a) Suppose $\mathrm{Gal}(K/\mathbf{Q}) = Q_8$ (the quaternion group of order 8). What are the possibilities for the degree of f?

   (b) Suppose $\mathrm{Gal}(K/\mathbf{Q}) = D_8$ (the dihedral group of order 8). What are the possibilities for the degree of f?

9. Prove there are no simple groups of order 132.

10. (a) Find all positive integers which occur as the order of some element of $\mathrm{GL}_2(\mathbf{Q})$. Exhibit an element of $\mathrm{GL}_2(\mathbf{Q})$ of order 3.

    (b) Find all positive integers which occur as the order of some element of $\mathrm{GL}_2(\mathbf{R})$. Exhibit an element of $\mathrm{GL}_2(\mathbf{R})$ of order 11.

11. Let R be the ring $\mathbf{Z}[\sqrt{10}]$.

    (a) Show that $13R$ is not a prime ideal of R.

    (b) Show that $17R$ is a prime ideal of R.