Algebra Qualifying Exam, UCI

June 20, 2005

Fully justify all your answers. (You may state and use standard big theorems.) Do as many problems as you can, as completely as you can. The exam is two and one-half hours.
Notation: Let $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{C}$ denote the rings of integers, rational numbers, and complex numbers, respectively. If R is a ring and n is a positive integer, then $\mathrm{GL}_n(R)$ is the group of invertible $n \times n$ matrices with entries in R.

(5 points) 1. Let $\mathbb{C}^*$ be the group of non-zero complex numbers under multiplication. Let H_n be the subgroup of n-th roots of unity. Show that the quotient group $\mathbb{C}^*/H_n$ is isomorphic to $\mathbb{C}^*$ by giving an explicit isomorphism.

(5 points) 2. Suppose G is a group of order n and F is a field. Prove that G is isomorphic to a subgroup of $\mathrm{GL}_n(F)$.

(6 points) 3. Let $\mathbb{F}_p$ denote the finite field of p elements. Decide if each of the following rings is a field.

(a) $\mathbb{F}_2[x]/(x^3 + x + 1)$

(b) $\mathbb{F}_3[x]/(x^3 + x + 1)$

(6 points) 4. Let R be the ring $\mathbb{Z}[\sqrt{-5}]$.

(a) Show that R is not a UFD.

(b) Factor the principal ideal (6) into a product of prime ideals in the ring R.

(12 points) 5. Classify the groups of order 12, up to isomorphism.

(10 points) 6. Let M be a matrix over $\mathbb{Q}$ with characteristic polynomial $(x + 1)^2x^4$ and minimal polynomial $(x + 1)^2x^2$.

(a) Find $\mathrm{trace}(M)$ and $\det(M)$.

(b) How many distinct conjugacy classes of such matrices are there in $\mathrm{GL}_6(\mathbb{Q})$? Explain.

(c) Write down a 6×6 matrix with entries in $\mathbb{Q}$ having the above characteristic and minimal polynomials.

(10 points) 7. Suppose p is a prime number and L/K is a field extension of degree p.

(a) Prove that if $K = \mathbb{Q}$, then L/K is separable.

(b) Prove that if $K = \mathbb{F}_p$, then L/K is separable.

(c) Give an example of a field extension L/K of degree p that is *not* separable.

(13 points) 8. Let K be the splitting field over $\mathbb{Q}$ of $x^8 - 1$.

(a) Find $[K : \mathbb{Q}]$.

(b) Describe the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$, both as an abstract group and as a set of automorphisms.

(c) Find explicitly all subgroups of G and the corresponding subfields of K under the Galois correspondence.

(15 points) 9. Suppose $f(x) \in \mathbb{Z}[x]$ is a polynomial of degree 5. Consider the following statements.

(i) f has no roots in $\mathbb{Q}$,

(ii) $f \equiv g_2 g_3 \pmod{11}$ where $g_2, g_3 \in (\mathbb{Z}/11\mathbb{Z})[x]$ are irreducible polynomials of degrees 2 and 3, respectively,

(iii) $f \equiv h_1 h_4 \pmod{17}$ where $h_1, h_4 \in (\mathbb{Z}/17\mathbb{Z})[x]$ are irreducible polynomials of degrees 1 and 4, respectively.

For each of the following assertions, either prove it is true or give a counterexample to show that it is false.

(a) If (i) holds then f is irreducible in $\mathbb{Q}[x]$.

(b) If (ii) holds then f is irreducible in $\mathbb{Q}[x]$.

(c) If (iii) holds then f is irreducible in $\mathbb{Q}[x]$.

(d) If both (i) and (ii) hold then f is irreducible in $\mathbb{Q}[x]$.

(e) If both (i) and (iii) hold then f is irreducible in $\mathbb{Q}[x]$.

(f) If both (ii) and (iii) hold then f is irreducible in $\mathbb{Q}[x]$.

(18 points) 10. Determine whether each of the following statements is true or false, and justify your answer with a proof or counterexample (justify your counterexample).

(a) The groups $\mathbb{Z}/20\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ and $\mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$ are isomorphic.

(b) The group of units in $\mathbb{Z}/12\mathbb{Z}$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$.

(c) Every UFD is a PID.

(d) For every commutative ring R, every subring of R is an ideal of R.

(e) For every commutative ring R, every ideal of R is a subring of R.

(f) For every commutative ring R with unity, every prime ideal of R is a maximal ideal of R.