Fully justify all your answers. (You may state and use standard big theo-
rems.) Do as many problems as you can, as completely as you can. The
exam is two and one-half hours.

Notation: Let $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{C}$ denote the rings of integers, rational numbers,
and complex numbers, respectively. If $R$ is a ring and $n$ is a positive integer,
then $\text{GL}_n(R)$ is the group of invertible $n \times n$ matrices with entries in $R$.

(5 points) 1. Let $\mathbb{C}^*$ be the group of non-zero complex numbers under multiplication.
Let $H_n$ be the subgroup of $n$-th roots of unity. Show that the quotient
group $\mathbb{C}^*/H_n$ is isomorphic to $\mathbb{C}^*$ by giving an explicit isomorphism.

(5 points) 2. Suppose $G$ is a group of order $n$ and $F$ is a field. Prove that $G$ is
isomorphic to a subgroup of $\text{GL}_n(F)$.

(6 points) 3. Let $\mathbb{F}_p$ denote the finite field of $p$ elements. Decide if each of the
following rings is a field.
(a) $\mathbb{F}_2[x]/(x^3 + x + 1)$
(b) $\mathbb{F}_3[x]/(x^3 + x + 1)$

(6 points) 4. Let $R$ be the ring $\mathbb{Z}[\sqrt{-5}]$.
(a) Show that $R$ is not a UFD.
(b) Factor the principal ideal $(6)$ into a product of prime ideals in the
ring $R$.

(12 points) 5. Classify the groups of order 12, up to isomorphism.

(10 points) 6. Let $M$ be a matrix over $\mathbb{Q}$ with characteristic polynomial $(x + 1)^2x^4$
and minimal polynomial $(x + 1)^2x^2$.
(a) Find $\text{trace}(M)$ and $\text{det}(M)$.
(b) How many distinct conjugacy classes of such matrices are there in
$\text{GL}_6(\mathbb{Q})$? Explain.
(c) Write down a $6 \times 6$ matrix with entries in $\mathbb{Q}$ having the above
characteristic and minimal polynomials.

(10 points) 7. Suppose $p$ is a prime number and $L/K$ is a field extension of degree $p$.
(a) Prove that if $K = \mathbb{Q}$, then $L/K$ is separable.
(b) Prove that if $K = \mathbb{F}_p$, then $L/K$ is separable.
(c) Give an example of a field extension $L/K$ of degree $p$ that is not
separable.
8. Let $K$ be the splitting field over $\mathbb{Q}$ of $x^8 - 1$.

(a) Find $[K : \mathbb{Q}]$.

(b) Describe the Galois group $G = \text{Gal}(K/\mathbb{Q})$, both as an abstract group and as a set of automorphisms.

(c) Find explicitly all subgroups of $G$ and the corresponding subfields of $K$ under the Galois correspondence.

9. Suppose $f(x) \in \mathbb{Z}[x]$ is a polynomial of degree 5. Consider the following statements.

(i) $f$ has no roots in $\mathbb{Q}$,

(ii) $f \equiv g_2g_3 \pmod{11}$ where $g_2, g_3 \in (\mathbb{Z}/11\mathbb{Z})[x]$ are irreducible polynomials of degrees 2 and 3, respectively,

(iii) $f \equiv h_1h_4 \pmod{17}$ where $h_1, h_4 \in (\mathbb{Z}/17\mathbb{Z})[x]$ are irreducible polynomials of degrees 1 and 4, respectively.

For each of the following assertions, either prove it is true or give a counterexample to show that it is false.

(a) If (i) holds then $f$ is irreducible in $\mathbb{Q}[x]$.

(b) If (ii) holds then $f$ is irreducible in $\mathbb{Q}[x]$.

(c) If (iii) holds then $f$ is irreducible in $\mathbb{Q}[x]$.

(d) If both (i) and (ii) hold then $f$ is irreducible in $\mathbb{Q}[x]$.

(e) If both (i) and (iii) hold then $f$ is irreducible in $\mathbb{Q}[x]$.

(f) If both (ii) and (iii) hold then $f$ is irreducible in $\mathbb{Q}[x]$.

10. Determine whether each of the following statements is true or false, and justify your answer with a proof or counterexample (justify your counterexample).

(a) The groups $\mathbb{Z}/20\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ and $\mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$ are isomorphic.

(b) The group of units in $\mathbb{Z}/12\mathbb{Z}$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$.

(c) Every UFD is a PID.

(d) For every commutative ring $R$, every subring of $R$ is an ideal of $R$.

(e) For every commutative ring $R$, every ideal of $R$ is a subring of $R$.

(f) For every commutative ring $R$ with unity, every prime ideal of $R$ is a maximal ideal of $R$. 

2