ALGEBRA QUALIFYING EXAM

September, 2006

(1) Short Answer:

(1a) (2 points) Define “prime ideal”:

(1b) (2 points) Define “Sylow p-subgroup”:

(1c) (3 points) Give an example of a unique factorization domain that is not a principal ideal domain.

(1d) (3 points) Give an example of a commutative ring R with identity, and a prime ideal M of R that is not a maximal ideal of R.

(2) (10 points) Let L be the splitting field of $x^3 - 2$ over $\mathbb{Q}$.

(a) Find $[L : \mathbb{Q}]$.

(b) Describe the Galois group $\mathrm{Gal}(L/\mathbb{Q})$, both as an abstract group and as a set of automorphisms.

(c) Find explicitly all subgroups of $\mathrm{Gal}(L/\mathbb{Q})$ and the corresponding subfields of L under the Galois correspondence.

(3) (10 points) Suppose n is a positive integer, and suppose A and B are two matrices in $M_{n\times n}(\mathbb{C})$ such that $AB = BA$. Prove that A and B have a common eigenvector.

(4) (10 points) Suppose G is a group and H is a finite normal subgroup of G. If G/H has an element of order n, prove that G has an element of order n.

(5) (10 points) Let C denote the center of $GL_2(\mathbb{F}_3)$ and let $PGL_2(\mathbb{F}_3) = GL_2(\mathbb{F}_3)/C$.

(a) Prove that $C = \{\pm I\}$ where I is the identity matrix.

(b) Prove that $PGL_2(\mathbb{F}_3) \simeq S_4$.

(6) (10 points) Describe the quotient ring $\mathbb{R}[x]/(x^2 + ax + b)$ in terms of $a, b \in \mathbb{R}$.

(7) (10 points) Let G be a finite group and suppose that p^n divides $|G|$, where p is a prime and n is a positive integer. Prove that G has a subgroup of order p^n. (You are allowed to use Sylow theorems without proving them, here.)

(8) (10 points) Suppose that H and K are subgroups of a group G, and suppose that H and K have finite index in G. Show that the intersection $H \cap K$ also has finite index in G.

(9) (10 points) Describe the conjugacy classes of $GL_2(\mathbb{C})$.

(10) (10 points) Suppose that p is a prime and M is an $\mathbb{F}_p[X]$-module. Suppose that $(X-1)^3 M = 0$ and $|(X-1)^2 M| = p$ and $|(X-1)M| = p^3$ and $|M| = p^7$. Determine M as an $\mathbb{F}_p[X]$-module, up to isomorphism.