(1) Short Answer:
   (1a) (2 points) Define “prime ideal”:
   (1b) (2 points) Define “Sylow p-subgroup”:
   (1c) (3 points) Give an example of a unique factorization domain that is not a principal ideal domain.
   (1d) (3 points) Give an example of a commutative ring \( R \) with identity, and a prime ideal \( M \) of \( R \) that is not a maximal ideal of \( R \).

(2) (10 points) Let \( L \) be the splitting field of \( x^3 - 2 \) over \( \mathbb{Q} \).
   (a) Find \([L : \mathbb{Q}]\).
   (b) Describe the Galois group \( \text{Gal}(L/\mathbb{Q}) \), both as an abstract group and as a set of automorphisms.
   (c) Find explicitly all subgroups of \( \text{Gal}(L/\mathbb{Q}) \) and the corresponding subfields of \( L \) under the Galois correspondence.

(3) (10 points) Suppose \( n \) is a positive integer, and suppose \( A \) and \( B \) are two matrices in \( M_{n \times n}(\mathbb{C}) \) such that \( AB = BA \). Prove that \( A \) and \( B \) have a common eigenvector.

(4) (10 points) Suppose \( G \) is a group and \( H \) is a finite normal subgroup of \( G \). If \( G/H \) has an element of order \( n \), prove that \( G \) has an element of order \( n \).

(5) (10 points) Let \( C \) denote the center of \( GL_2(\mathbb{F}_3) \) and let \( PGL_2(\mathbb{F}_3) = GL_2(\mathbb{F}_3)/C \).
   (a) Prove that \( C = \{ \pm I \} \) where \( I \) is the identity matrix.
   (b) Prove that \( PGL_2(\mathbb{F}_3) \simeq S_4 \).

(6) (10 points) Describe the quotient ring \( \mathbb{R}[x]/(x^2 + ax + b) \) in terms of \( a, b \in \mathbb{R} \).

(7) (10 points) Let \( G \) be a finite group and suppose that \( p^n \) divides \( |G| \), where \( p \) is a prime and \( n \) is a positive integer. Prove that \( G \) has a subgroup of order \( p^n \). (You are allowed to use Sylow theorems without proving them, here.)

(8) (10 points) Suppose that \( H \) and \( K \) are subgroups of a group \( G \), and suppose that \( H \) and \( K \) have finite index in \( G \). Show that the intersection \( H \cap K \) also has finite index in \( G \).

(9) (10 points) Describe the conjugacy classes of \( GL_2(\mathbb{C}) \).

(10) (10 points) Suppose that \( p \) is a prime and \( M \) is an \( \mathbb{F}_p[X] \)-module. Suppose that \( (X - 1)^3 M = 0 \) and \( |(X - 1)^2 M| = p \) and \( |(X - 1) M| = p^3 \) and \( |M| = p^7 \). Determine \( M \) as an \( \mathbb{F}_p[X] \)-module, up to isomorphism.