Comprehensive Exam in Analysis Spring 2009 – Summary of the Problems

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies the equation $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Prove the following statements:

(a) If f is positive at one point of $\mathbb{R}$, then f is positive at every point of $\mathbb{R}$.

(b) If f is differentiable at one point of $\mathbb{R}$, then f is differentiable at every point of $\mathbb{R}$.

(2) Let $f_n, n = 1, 2, ...$ and f be Riemann integrable real-valued functions defined on $[0, 1]$. For each of the following statements, determine whether the statement is true or not; prove your claims:

(a) If $\lim_{n\to\infty} \int_0^1 |f_n(x) - f(x)|dx = 0$, then $\lim_{n\to\infty} \int_0^1 |f_n(x) - f(x)|^2 dx = 0$.

(b) If $\lim_{n\to\infty} \int_0^1 |f_n(x) - f(x)|^2 dx = 0$, then $\lim_{n\to\infty} \int_0^1 |f_n(x) - f(x)|dx = 0$.

(3) Suppose that $f : \mathbb{R}^3 \to \mathbb{R}^3$ is the vector-valued function defined by

$$f(x, y, z) = (x + y^2 + 100z, x + 3y - 100z, e^{-z+100y^2}) \text{ for all } (x, y, z) \text{ in } \mathbb{R}^3.$$

(a) Compute the determinant of the Jacobian matrix of f at the point $(0, 0, 0)$.

(b) Is there an open neighborhood of $(0, 0, 0)$ such that f is one-to-one in this neighborhood? If your answer is 'Yes', please give the reason and find an explicit example of an open neighborhood of $(0, 0, 0)$ on which f is one-to-one.

(4) Let $f : [a, b] \to \mathbb{R}$ be a real-valued function defined on a closed bounded interval $[a, b]$ in $\mathbb{R}$. We define the **graph** of the function f to be set Γ_f of points (x, y) in $\mathbb{R}^2$ such that $x \in [a, b]$ and $y = f(x)$.

Prove or Disprove The function f is continuous on $[a, b]$ if, and only if, the set Γ_f is a closed subset of $\mathbb{R}^2$.

(5) Prove that the set $\mathbb{N}$ of all positive integers has the same cardinality as the Cartesian product $\mathbb{N}\times\mathbb{N}$. (As usual, $\mathbb{N}$ is the set of all positive integers. Also, to say that a set X has the same cardinality as a set Y means that there is a one-to-one map $\varphi : X \to Y$ of X onto Y.)

NOTE: In order to get full credit for this problem, you should construct an explicit example of a one-to-one map of $\mathbb{N}$ onto $\mathbb{N}\times\mathbb{N}$.

(6) Suppose that $f : [a, b] \to \mathbb{R}$ is continuously differentiable on the closed interval $[a, b]$ in $\mathbb{R}$, and suppose that $g : [a, b] \to \mathbb{R}$ is a monotonic function such that $g(a) = f(a)$ and $g(b) = f(b)$.

Prove or Disprove: There exists a constant M such that $|f(x) - g(x)| \leq M \cdot |b - a|$ for all x in $[a, b]$.

(7) Define a sequence $(a_1, a_2, \ldots a_n, \ldots)$ recursively by setting $a_1 = 1$, $a_2 = 3$, and $a_{n+2} = (a_{n+1} + 2a_n)/3$ for $n \geq 1$. Prove that the sequence (a_n) converges, and compute its limit.

(8) Let f be a real-valued function defined on the real line.

(a) Prove or Disprove If f is uniformly continuous on $\mathbb{R}$ then f^2 is uniformly continuous on $\mathbb{R}$.

(b) Prove or Disprove If f is uniformly continuous on $\mathbb{R}$ then $f^2/(1 + f^2)$ is unifomly continuous on $\mathbb{R}$.

(9) Suppose f is a Riemann integrable function on $[0, 1]$. Prove that $\lim_{n\to\infty} \int_0^1 x^n f(x)\, dx = 0$.