Algebra Qualifying Exam, June 2009 (10 points each problem)

1. Let $D_{2n}$ be the dihedral group of order $2n$.
   (a) Prove that if $p$ is an odd prime, then a Sylow $p$-subgroup of $G$ is normal and cyclic.
   (b) Prove that if $2n = 2^a \cdot k$ where $k$ is odd then the number of Sylow 2-subgroups of $D_{2n}$ is $k$. Describe all these subgroups.

2. Let $G$ be a group such that $\text{Aut}(G)$ is cyclic. Show that $G$ is abelian.

3. Let $\mathbb{Z}$ be the ring of integers, $\mathbb{F}_5$ be the field with five elements.
   (a) Determine whether the rings $\mathbb{F}_5[x]/(x^2 + 1)$ and $\mathbb{F}_5[x]/(x^2 + 2)$ are isomorphic.
   (b) List all ideals in the ring $\mathbb{Z}[x]/(2, x^3 + 1)$.

4. Prove that the Galois group of the polynomial $x^5 - 2$ is isomorphic to the group of all matrices of the form
   $$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$
   where $a, b \in \mathbb{F}_5$ and $a \neq 0$.

5. Let $F$ be a field of characteristic not dividing $n$. Show that the matrix equation $XY - YX = I_n$ has no solutions, where $X$ and $Y$ are unknown $n \times n$ matrices with entries in $F$ and $I_n$ is the identity matrix.

6. Let $T$ be a linear operator on a finite dimensional vector space $V$ over $\mathbb{Q}$ such that $T^{15} = I$. Assume that both $T^3$ and $T^5$ have no non-zero fixed points in $V$. Show that the dimension of $V$ is divisible by 8.

7. Let $A$ be a finite Abelian group, $p$ be a prime dividing $|A|$ and $k$ be largest such that $p^k$ divides $|A|$. Prove that $\mathbb{Z}/p^k \mathbb{Z} \otimes A$ is isomorphic to the Sylow $p$-subgroup of $A$.

8. Consider complex representations of the finite group $S_4$ up to isomorphism.
   (a) Show that $S_4$ has exactly two one dimensional complex representations.
   (b) Prove that its other pairwise non-isomorphic complex representations have dimensions 2, 3, and 3.

9. Let $R$ be a commutative local ring with maximal ideal $M$.
   (a) Show that if $x \in M$, then $1 - x$ is invertible.
   (b) Show that if in addition that $R$ is Noetherian and $I$ is an ideal satisfying $I^2 = I$, then $I = 0$.

10. Let $\mathbb{F}_q$ be a finite field of q elements. Show that every element $x \in \mathbb{F}_q$ can be written as a sum of two squares in $\mathbb{F}_q$, that is, $x = y^2 + z^2$ for some $y, z \in \mathbb{F}_q$. 

1