Convex Optimization in Image Processing

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5-5-2010
Inverse Problems

Let $f$ be some given measurements of an image $h$

**Goal:** Find $h$ given $f$ and knowledge of the kinds of measurements taken

Some Examples:

- **Denoising:** $f = h + \text{noise}$
- **Deblurring:** $f = \text{blurred } h + \text{noise}$
- **Super Resolution:** $f = \text{low resolution } h + \text{noise}$
Variational Models

Model the problem by defining a function $F(u)$ on images $u$ such that

- $F(u)$ is large when $u$ is a poor solution
- $F(u)$ is small when $u$ is a good solution

**Data Fidelity:** $F(u)$ should be smaller when $u$ is consistent with the measurements $f$

**Regularization:** Data fidelity is usually not enough by itself. To make the problem well posed, add an assumption about $u$:

- smooth $u$
- piecewise constant $u$
- sparse $u$, etc...

$F(u)$ should be smaller when the assumption is better satisfied
Optimization

Represent images as real $M \times N$ matrices or as vectors in $\mathbb{R}^{MN}$.

Find $u^{*} \in \mathbb{R}^{MN}$ that minimizes $F(u)$

**Calculus Approach:** Solve $\nabla F(u^{*}) = 0$

**Iterative Approach:** Find $u^{k} \rightarrow u^{*} \quad k = 1, 2, 3, ...$

Issues:

- linear versus nonlinear
- differentiability
- number of variables
- constraints on $u$
- local versus global minima
- convex versus nonconvex
Convexity

In fact the great watershed in optimization isn’t between linearity and nonlinearity, but convexity and nonconvexity.

- R. T. Rockafellar

\[ F((1 - s)u_1 + su_2) \leq (1 - s)F(u_1) + sF(u_2) \quad \text{for all } u_1, u_2 \]
\[ 0 \leq s \leq 1 \]
Local Min versus Global Min

If $F$ is convex, Local Minimum $\Rightarrow$ Global Minimum

Smooth, Nonconvex

Convex, Nonsmooth

Steepest descent, "go downhill", finds local min

Proximal point method finds global min

Proximal point method diagram from Bertsekas and Tsitsiklis
Solving Convex Problems

- There are efficient algorithms for convex optimization
- Image processing problems modeled as convex optimization problems can be reliably solved

Deblurring Example: $F(u)$ was defined to be a convex function that encourages data fidelity and prefers piecewise constant $u$

Original image  Blurry/Noisy  Recovered

(This was solved using an iterative method that is a generalization of the proximal point method.)
Nonconvex Problems

- Nonconvex problems are much harder in general

Example: Blind Deblurring (don’t know how image was blurred)
Example: Registration

Want to align $h_1$ and $h_2$

$$F(v) = \int (h_1(x+v(x)) - h_2(x))^2 \, dx$$

$h_1(x)$ and $h_2(x)$

$F(v)$ in the case when $v(x) = \tau$

(translation)
Convex Approximation

- Approaches for nonconvex problems often require convex optimization for subproblems
- Sometimes can approximate a nonconvex model by a convex one

Convex image registration example:

The good: based on convex model, so can find global minimum

The bad: slow to compute minimizer due to many extra variables (had to essentially double dimensionality of problem to make it convex)