Convex Optimization in Image Processing

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Inverse Problems

Let f be some given measurements of an image h

Goal: Find h given f and knowledge of the kinds of measurements taken

Some Examples:

- Denoising: $f = h$ + noise
- Deblurring: $f =$ blurred h + noise
- Super Resolution: $f =$ low resolution h + noise

Variational Models

Model the problem by defining a function $F(u)$ on images u such that

- $F(u)$ is large when u is a poor solution
- $F(u)$ is small when u is a good solution

Data Fidelity: $F(u)$ should be smaller when u is consistent with the measurements f

Regularization: Data fidelity is usually not enough by itself. To make the problem well posed, add an assumption about u:

- smooth u
- piecewise constant u
- sparse u, etc...

$F(u)$ should be smaller when the assumption is better satisfied

Optimization

Represent images as real $M \times N$ matrices or as vectors in $\mathbb{R}^{MN}$.

Find $u^* \in \mathbb{R}^{MN}$ that minimizes $F(u)$

Calculus Approach: Solve $\nabla F(u^*) = 0$

Iterative Approach: Find $u^k \to u^* \qquad k = 1, 2, 3, ...$

Issues:

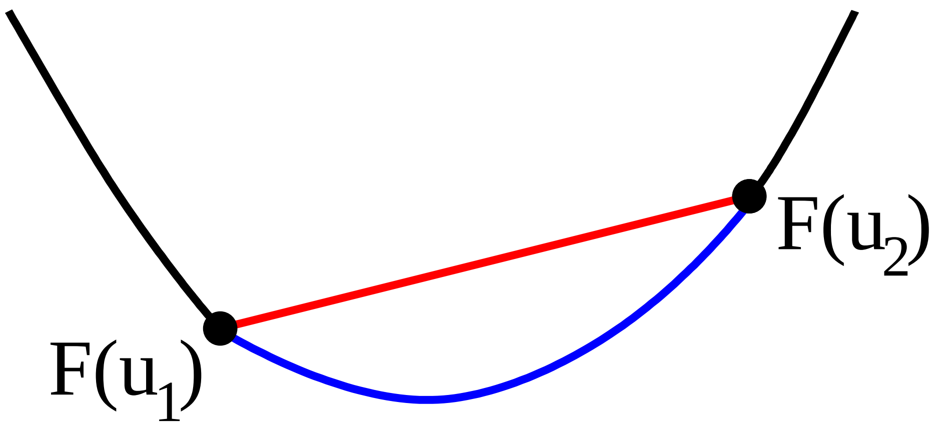
- linear versus nonlinear
- differentiability
- number of variables
- constraints on u
- local versus global minima
- convex versus nonconvex

Convexity

In fact the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity.

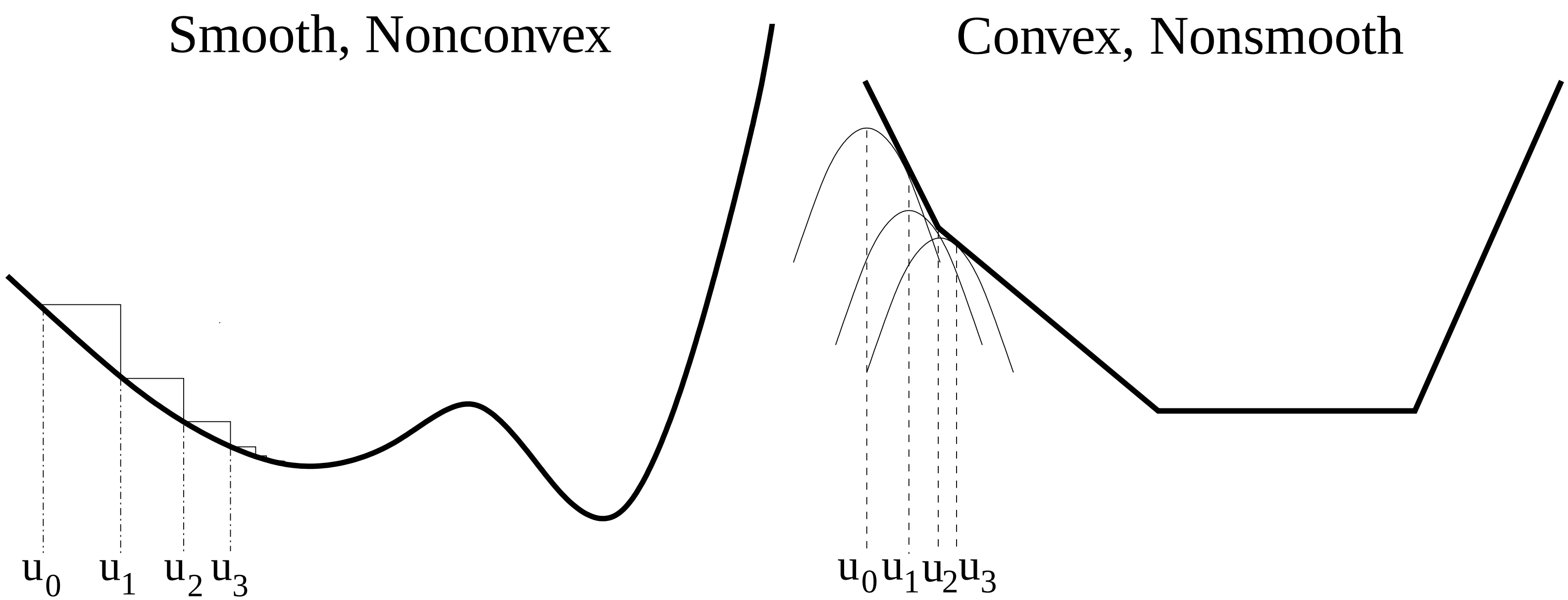
- R. T. Rockafellar

$$F((1-s)u_1 + su_2) \leq (1-s)F(u_1) + sF(u_2) \qquad \text{for all } u_1, u_2$$
$$0 \leq s \leq 1$$



Local Min versus Global Min

If F is convex, Local Minimum $\Rightarrow$ Global Minimum



Steepest descent,
"go downhill",
finds local min

Proximal point method
finds global min

Proximal point method diagram from Bertsekas and Tsitsiklis

Solving Convex Problems

- There are efficient algorithms for convex optimization
- Image processing problems modeled as convex optimization problems can be reliably solved

Deblurring Example: $F(u)$ was defined to be a convex function that encourages data fidelity and prefers piecewise constant u

Original image

Blurry/Noisy

Recovered

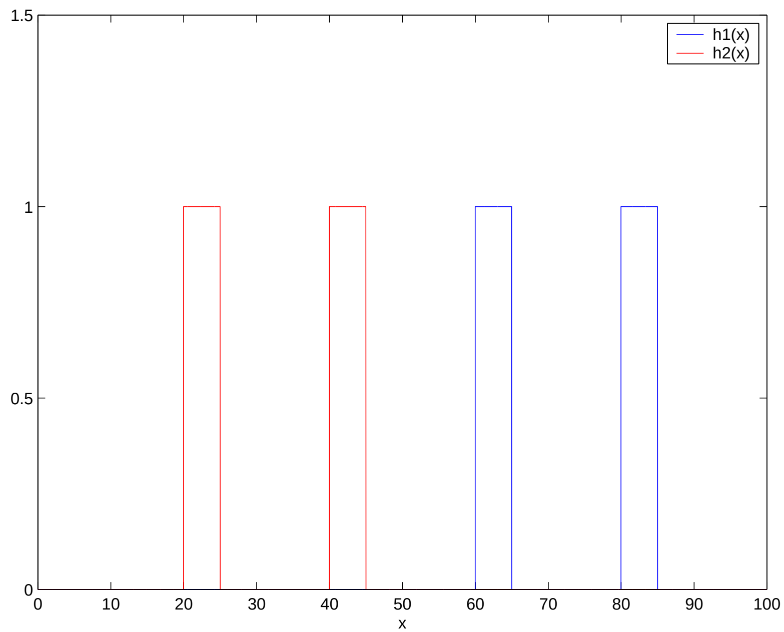
(This was solved using an iterative method that is a generalization of the proximal point method.)

Nonconvex Problems

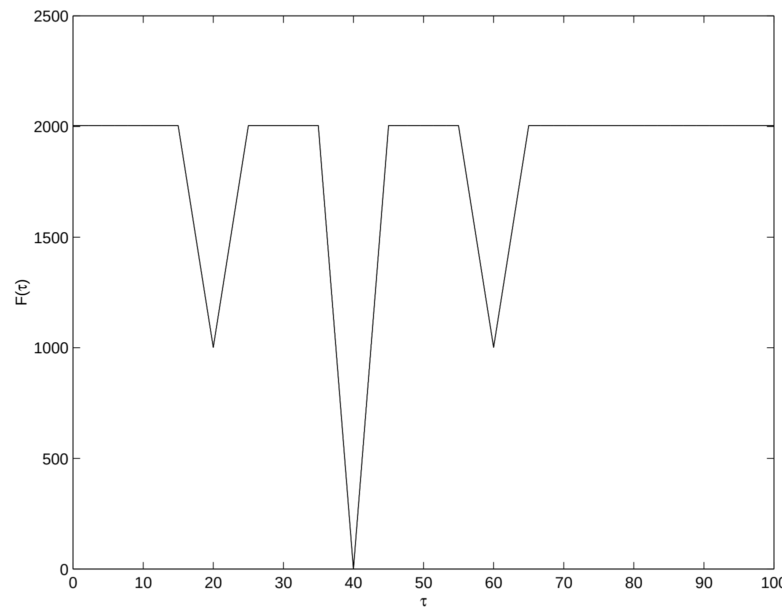
- Nonconvex problems are much harder in general

Example: Blind Deblurring (don't know how image was blurred)
Example: Registration

Want to align h_1 and h_2 $$F(v) = \int (h_1(x+v(x)) - h_2(x))^2 dx$$



$h_1(x)$ and $h_2(x)$

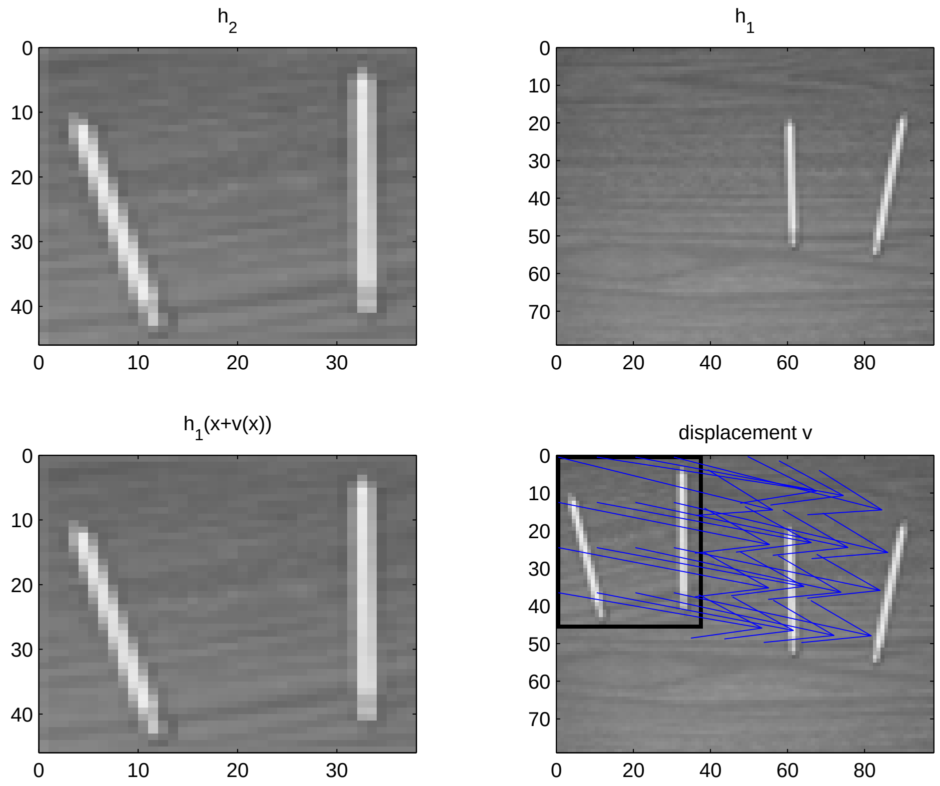


$F(v)$ in the case when $v(x) = \tau$ (translation)

Convex Approximation

- Approaches for nonconvex problems often require convex optimization for subproblems
- Sometimes can approximate a nonconvex model by a convex one

Convex image registration example:



The good: based on convex model, so can find global minimum
The bad: slow to compute minimizer due to many extra variables
(had to essentially double dimensionality of problem to make it convex)