Combined Games

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August 17, 2013

Abstract

What happens when you play Chess and Tic-Tac-Toe at the same time? What about Mancala, Othello and Checkers? Playing multiple games together as a Combined Game introduces an entirely new, infinite set of games to analyze and play. We use traditional Game Theory tools to examine combined pairs of small, well-solved games, yielding information on the optimal strategies for playing each combined game. The results of our analysis reveal patterns for two Normal Games or any amount of Meiseré Games being played as one Combined Game and suggest that such patterns exist when combining other classes of games.

1 Introduction

Let us begin by introducing and defining the notion of a *Combined Game*.

1.1 What are *Combined Games*?

A *Combined Game* G is defined as the following:

Definition 1. *A* Combined Game G *is a finite set of* n *sub-games, where* $n \geq 2$.

We assume every $g \in G$ has the following properties:

- Two Player,
- Sequential Play,
- *Well-Solved/Well-Studied.*

1.2 Rules of *Combined Games*

In addition to the above properties, a Combined Game G has the following rules:

1. On a given turn, a player may make one legal move on any $g \in G$.
2. A player in a Combined Game G must skip their turn if and only if for every $g \in G$, the current player has no legal moves on g.
3. A player may forfeit one $g \in G$ if and only if that player forfeits all $h \in G$, where h is a sub-game of G.

1.3 Example of Playing a Combined Game

Here is an example of a Combined Game of Hexapawn and Tic-Tac-Toe.

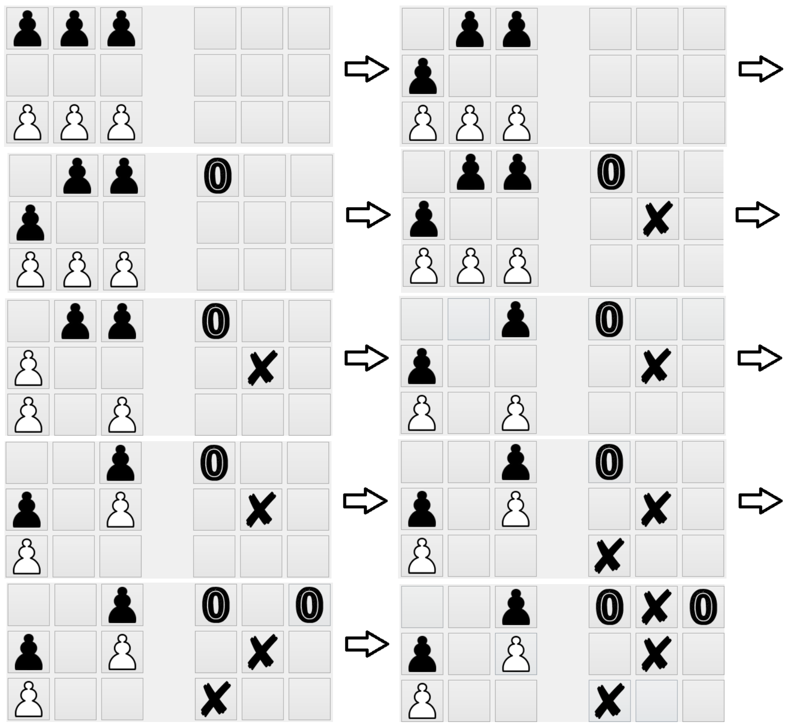

Figure 1: *In this example, each arrow indicates the resulting board from a player moving. The first board is the opening board, then the next board is the result of Player 1 moving on Hexapawn, and the next board is the result of Player 2 moving on Tic-Tac-Toe, and so on and so forth.*

2 Game Theory Notations and Definitions

This section outlines some important Game Theory definitions and notations that are important to our analysis of Combined Games.

Winning Strategy A *winning strategy* of a game is a strategy s such that if a player follows s, that player wins the game.

N-Position A board is said to be an *N-Position* if the Next Player to make a move has the winning strategy.

P-Position A board is said to be a *P-Position* if the Previous Player to move has the winning strategy.

Optimal Play A player plays a game *optimally* if they try to win as best they can given the available information. [2, pg. 3]

3 Games Analyzed in Various Combinations

To begin analyzing the nature of Combined Games, we start by picking combinations of small and simple games. We briefly describe each game analyzed and state the which player has the winning strategy. [1]

3.1 Game 1: Hexapawn

We focus on the 3x3 case. Players each have three chess pawns, and move them according to the rules of chess. A player wins by moving their pawn to the board side opposite their own or by making the last legal move. For the 3x3 case, assuming optimal play, it is a P-Position. See Figure 2 in Section 3.5 for an example of playing Hexapawn.

3.2 Game 2: Tic-Tac-Toe

Players alternate drawing an 'X' or an 'O' on a 3x3 board. The player who draws three of their symbol in a row is the winner. Assuming optimal play, this game always ends in a tie.

3.3 Game 3: Poison

Played with a pile of objects greater than 0, players alternate removing one or two objects from the top of the pile. The player who removes the last object loses. Given a pile of size X, the position of the game is:

- an N-Position if $X \equiv 0, 2 \ (mod\ 3)$.
- a P-Position if $X \equiv 1 \ (mod\ 3)$.

See Figure 3 in Section 3.5 for an example of a game of Poison.

3.4 Game 4: Cupcake

Played exactly the same as Poison, except the player that removes the last object wins. Given a pile of size M, the position of the game is:

- an N-Position if $M \equiv 1, 2 \ (mod\ 3)$.
- a P-Position if $M \equiv 0 \ (mod\ 3)$.

See Figure 4 in Section 3.5 for an example of a game of Cupcake.

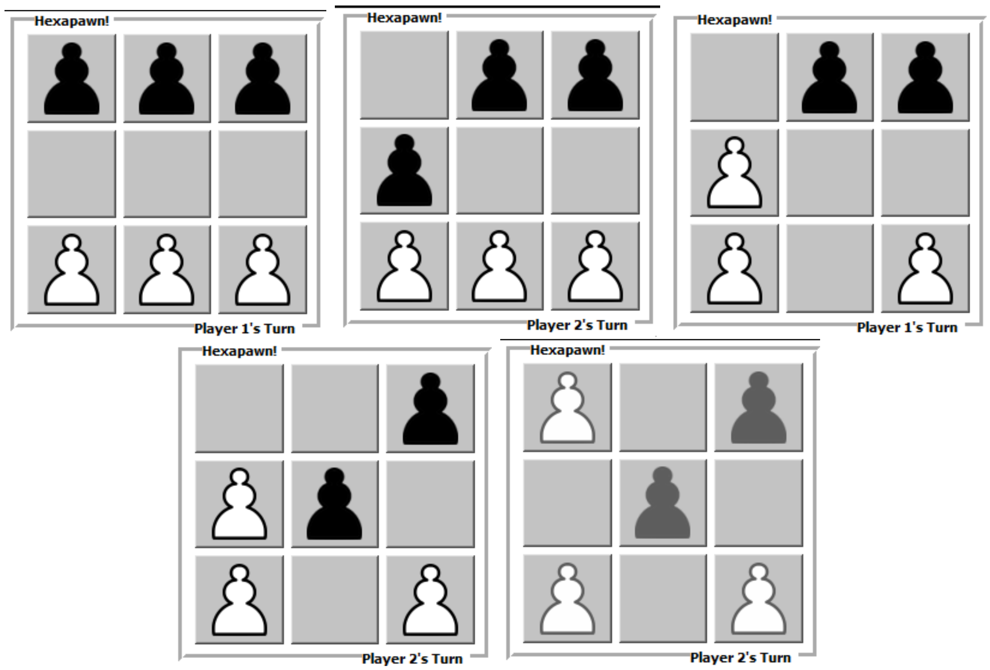

Figure 2: *This figure outlines a game of Hexapawn.*

3.5 Examples of the Outlined Games

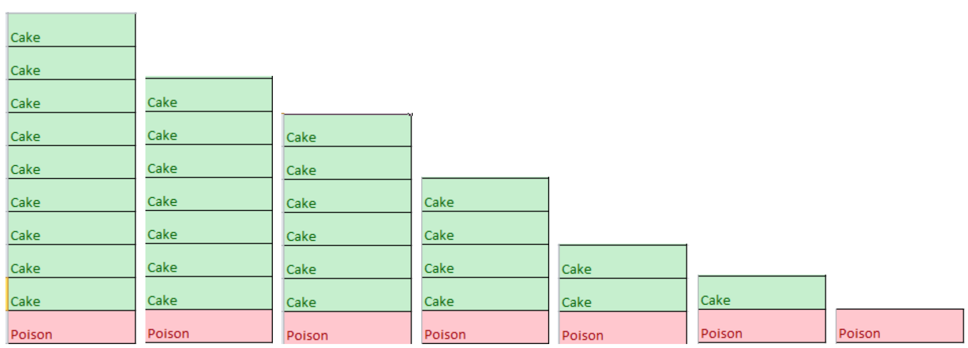

Figure 3: *This figure outlines a game of Poison of size 10.*

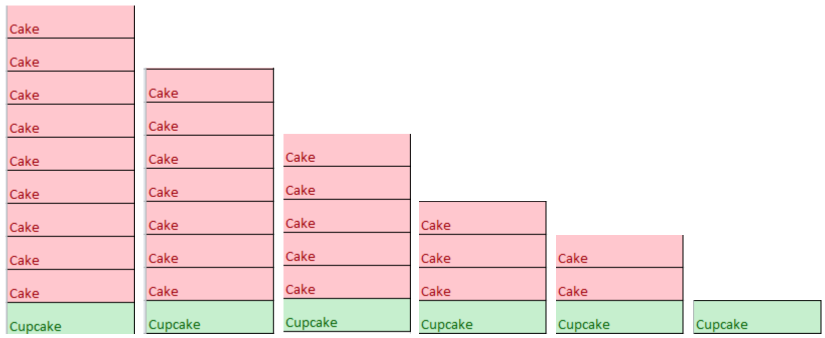

Figure 4: *This figure outlines a 10-Piece game of Cupcake.*

4 Important Properties of Combined Games

In our analysis of Combined Games, we found two properties that are essential to the optimal play on a Combined Game.

The Tic-Tac-Toe Effect

Principle 1. *A game G is said to have the* Tic-Tac-Toe Effect *(TTTE) if:*

1. *G optimally ends in a draw*
2. *The first player to move on G is the last player to move on G.*

The Skip-and-Lose Principle

Principle 2. *Given a game board g of a game G, if:*

- *Player A makes a move on board g, which results in board g',*
- *Player B skips making a move on board g',*
- *Player A make a move on board g', resulting in board g",*

and all resulting boards from g" result in a loss for Player B, then g is said to have the Skip-and-Lose Principle *(SLP).*

5 Theorems and Proofs

Here, we present the combinations of games we have examined, conjectures about those combinations and proofs of those conjectures, and some other conjectures and proofs as well.

5.1 Combination One: Hexapawn and Tic-Tac-Toe

Theorem 1. *In the Combined Game of Hexapawn and Tic-Tac-Toe, Player 1 has the winning strategy, which is to play on Tic-Tac-Toe first, then play on whatever board Player 2 moves on.*

Proof. There are two things that must be proven: 1. That there exists a winning strategy. 2. The winning strategy belongs to Player 1, and that strategy is to play on Tic-Tac-Toe first, then to move on whatever board Player 2 moves on.

1. Because Hexapawn cannot end in a tie, and Tic-Tac-Toe optimally ends in a draw, at most 1 player will receive 1 point. Thus, Hexapawn & Tic-Tac-Toe (H&TTT) cannot end in a tie. Because H&TTT cannot end in a tie and is a progressively bounded combinatorial game, there exists a winning strategy.

2. Assuming optimal play, the opening move on H&TTT must be on the Tic-Tac-Toe board because the first move on Hexapawn is a P-Position. Thus Player 1 moves on Tic-Tac-Toe first. Now, by the Tic-Tac-Toe Effect, if Player 2 continues playing Tic-Tac-Toe, Player 2 is the player to make the first move on Hexapawn, which guarantees Player 2's loss. If Player 2 moves on Hexapawn, and if Player 1 follows and also moves on Hexapawn, Player 2 loses H&TTT since Tic-Tac-Toe ends in a tie and Player 1 wins Hexapawn. But, if Player 2 moves on Hexapawn, then Player 1 moves on Tic-Tac-Toe, then Player 2 moves on Hexapawn a second time, by the Skip-And-Lose Principle, Player 2 wins Hexapawn and Player 1 wins Tic-Tac-Toe, which leaves H&TTT as a tie game. However, since Player 1 plays optimally, if Player 2 moves on Hexapawn, Player 1 follows. Additionally, if Player 2 moves on Tic-Tac-Toe, Player 1 follows because of optimal play. Thus, Player 1 has the winning strategy.

□

5.2 Combination Two: Hexapawn and Poison

Theorem 2. *The Combined Game of Hexapawn and Poison (of size $n > 0$) always ends in a draw.*

Proof. First, if one player has a winning strategy, playing the sub-games simultaneously has the same effect as playing the two sub-games one by one. Assume that Player A has a winning strategy. If Player A follows his winning strategy, he wins both sub-games because neither Poison or Hexapawn is able to draw and if Player A only wins only one game then he must lose the other. Since player A has winning strategy, he would not allow himself to skip one move in one sub-game. Furthermore, Player A is always able to prevent himself from skipping one move in one game by choosing the game Player B moved on. As a whole, because Player A has the winning strategy, he chooses to play the game the Player 2 plays. In that case, playing the two sub-games simultaneously has the same effect as playing the two sub-games one by one.
Second, if one player has a winning strategy, he would force the other player to take both the first move in Hexapawn and the last move in poison. In a Hexapawn, the second player to move has winning strategy, and in Poison, the player who takes the last move loses. If Player A has winning strategy, he would win both the games. Moreover, since playing the two sub-game simultaneously has the same effect as playing the two games one by one; if one player makes the first move in Hexapawn, he would become Player 1 in Hexapawn forever and he would lose the sub-game. In that case, Player A would try to force the other player to take the first move in Hexapawn. To sum up, if Player A has winning strategy, he would force the other player to take both the first move in Hexapawn and the last move in Poison.
Finally, no one is able to force the other player to take both the first move in Hexapawn and the last move in poison. If both players play optimally, no one would like to take the first move in Hexapawn. The winning strategy holder must try and force the other player to take that move. However, the only way to force the other player to play first in Hexapawn is to take the last move in Poison. In another words, if the winning strategy holder wants to win the Hexapawn game, he has to take the poison piece in Poison. In that case he would lose Poison, but win Hexapawn. Thus, the combined game of Hexapawn and Poison ends in a draw. □

5.3 Combination Three: Tic-Tac-Toe and Poison

Theorem 3. *In the Combined Game of Tic-Tac-Toe and Poison of any size $n \geq 1$,*

- *If $n \equiv 0, 1\ (mod\ 3)$, the game is an N-Position.*
- *If $n \equiv 2\ (mod\ 3)$, the game is a P-Position.*

Proof. We proceed by induction, with base cases $n = 1$ and $n = 2$.

Case 1: $n = 1$: If $n = 1$, then Player 1 plays on Tic-Tac-Toe because moving on Poison yields a loss. The same follows for Player 2. By the Tic-Tac-Toe Effect, when Tic-Tac-Toe is finished, Player 2 must take from Poison, and thus Player 1 wins.

Case 2: $n = 2$: If $n = 2$, then Player 1 has two moves: take one from the Poison or move on Tic-Tac-Toe. If he moves on Poison, then this case is reduced to $n = 1$, and thus Player 2 wins. If Player 1 moves on Tic-Tac-Toe, Player 2 follows by the Skip-and-Lose Principle. Thus, when Tic-Tac-Toe is finished, Player 2 begins on Poison. He takes one from Poison, leaving Player 1 to lose Poison. Thus, Player 2 wins.

Inductive Hypothesis: Assume true for Poison of size n: Consider the Combined Game of Tic-Tac-Toe & Poison of size $n + 1 = k$. If $k \equiv 2\ (mod\ 3)$ then $(k - 1) \equiv 1\ (mod\ 3)$ and $(k - 2) \equiv 0\ (mod\ 3)$, which are N-Positions. Because Poison is a game such that every N-Position can move to at least one P-Position, k must be a P-Position. If $k \equiv 1\ (mod\ 3)$, then $(k - 1) \equiv 0\ (mod\ 3)$ and $(k - 2) \equiv 2\ (mod\ 3)$. Because $k - 1$ is an N-Position and $k - 2$ is a P-Position and because every P-Position can only move to an N-Position, k must be an N-Position. The argument is similar when $k \equiv 0\ (mod\ 3)$. Thus, we have shown which player has the winning strategy for $n + 1$.

□

5.4 Combination Four: Cupcake and Cupcake

Theorem 4. *In the Combined Game of Cupcake (of size $m > 0$) & Cupcake (of size $n > 0$), then the winning player follows the presented pattern:*

1. *If both piles are P-Positions, then the game is a P-Position.*
2. *If both piles are N-Positions, then the game is a draw.*
3. *If one pile is an N-Position, and one pile is a P-Position, then the game is an N-Position.*

The conjecture for Cupcake & Cupcake was derived from a pattern found in the table of different size game boards.

Game 1, Position	Game 2, Position	Overall Position
1,N	1,N	D
1,N	2,N	D
2,N	2,N	D
1,N	3,P	N
2,N	3,P	N
3,P	3,P	P
1,N	4,N	D

Table of Cupcake & Cupcake board positions.

This pattern continues up to two piles of 6, and from there the conjecture was made. However, Cupcake & Cupcake is just a specific case of Two Normal Games being played simultaneously. The conjecture and proof for Two Normal Games is presented in Section 5.5.

5.5 Combination Five: Two Normal Games

Theorem 5. *In a Combined Game G of two* Normal *sub-games, g_1 and g_2, such that from every N-Position in g_1 and g_2, there exists a move to another N-Position and g_1, g_2 cannot end in a draw, we have:*

1. *If g_1 and g_2 are P-Positions, then G is a P-Position.*
2. *Without loss of generality, if g_1 is an N-Position and g_2 is a P-Position, then G is an N-Position.*
3. *If g_1 and g_2 are N-Positions, then G is a draw.*

Proof. Let $g_1, g_2 \in G$.

1. Let g_1 and g_2 be P-Positions. WLOG, assume Player 1 moves on g_1, resulting in g_1', which is an N-Position. Thus, Player 2 moves g_1' to g_1", where g_1" is a P-Position. Thus Player 1 must move on g_1" or g_2, which are both P-Positions. Following this process, when g_1 is finished, Player 2 is the winner, and whatever state g_2 is currently in is a P-Position. Thus, Player 2 wins g_2. Because Player 2 wins both g_1 and g_2, G is a P-Position.

2. WLOG, let g_1 be an N-Position and g_2 be a P-Position. Then, assuming optimal play, Player 1 moves g_1 to g_1', where g_1' is a P-Position. Then, Player two must begin on g_1' and g_2, which are both P-Positions. Thus, as we have shown previously, Player 2 loses on both g_1' and g_2. Thus, G is an N-Position.

3. Let g_1 and g_2 be N-Positions. Then, moving on g_1 or g_2 results in g_1' and g_2'. WLOG, assume Player 1 plays optimally and moves g_1 to g_1'.

   Case 1: g_1' is a P-Position Let g_1' be a P-Position. As such, the combined game G' = $\{g_1$',$g_2\}$ is an N-Position, and thus Player 2 wins. However, because Player 1 moves optimally, he does not move g_1 to a P-Position.

   Case 2: g_1' is an N-Position Let g_1' be an N-Position. Now, Player 2 must move on g_1' or g_2. Assuming Player 2 plays optimally, he does not move g_1' or g_2 to a P-Position by the argument in Case 1. Thus, both Player 1 and Player 2 continuously reduce g_1 and g_2 to N-Positions. WLOG, assume Player 1 wins g_1. Then, the current state of g_2 is an N-Position, and thus Player 2 wins g_2. Because both players win a game, G ends in a draw.

□

5.6 Combination Six: Poison and Poison

For an easier understanding of Poison & Poison, assume the Combined Game is simply two piles of Poison, where a player can only take from one pile at a time.

Theorem 6. *The Combined Game of Poison (of size $m > 0$) & Poison (of size $n > 0$) always ends in a draw.*

Proof. Because Poison is a Meiseré game, optimal play assumes both players avoid picking the last item in a pile. Thus, if one pile in Poison & Poison has only one piece remaining, if there is more than one piece in the other pile, the players take pieces from that pile, until just 1 remains. Thus, each player must take the one poison from one of the piles. Therefore, each player receives 1 point, which means that Poison & Poison ends in a draw. □

5.7 Combination Seven: X Games of Poison

Theorem 7. *In the Combined Game of X games of Poison (X-Poison) of size $n_1, n_2, \cdots, n_X$ where $X \geq 2$ and $n_\imath \geq 1$ for $1 \leq \imath \leq X$:*

1. *If N is even, then X-Poison is a draw.*
2. *If N is odd, then the winner of X-Poison is dependent on the number of pieces p in the last pile with more than one piece*
   (a) *If $p \equiv 0, 2 \pmod 3$, then X-Poison is an N-Position.*
   (b) *If $p \equiv 1 \pmod 3$, then X-Poison is a P-Position.*

Proof. Because X-Poison is a Combined Game of M-Meiseré games, it follows from the proof of the M-Meiseré Combined Game that the above strategy is correct. □

5.8 Combination Eight: M Meiseré Games

Theorem 8. *In a Combined Game G of M* Meiseré *sub-games, such that each sub-game cannot end in a tie, the following is true:*

1. *If M is even, then G always ends in a draw,*
2. *If M is odd, then the winner of G depends on the status of the last sub-game b of G such that b is the only sub-game that is not one move away from a terminal position.*

Proof.
1. If M is even, we can examine the case of $M = 2$ first, with sub-games g_1 and g_2. In a Meiseré game, the optimal strategy is to avoid making the last move. Assume Player 1 and Player 2 play according to the optimal strategy. As such, let g_1 and g_2 be the starting positions of the two sub-games, and let g_1* and g_2* denote a game state of g_1 and g_2, respectively, that is one move away from a terminal position. After some rounds of play, WLOG assume g_1 has been moved to g_1*, and g_2 has been moved to some g_2' such that $g_2' \neq g_2^*$. Because Player 1 and Player 2 follow the optimal strategy, neither player moves on g_1*; each player moves on g_2', until g_2' is moved to g_2*. Because g_1 and g_2 cannot end in a tie, each player loses one of the sub-games, g_1 or g_2. Thus, G is a draw.
   It follows that if there is an even amount of M sub-games such that $M > 2$, this pattern holds; thus, G ends in a draw.
2. If M is odd, then we can reduce the game to the case of $M - 1$ Meiseré games and 1 Meiseré game that we denote as b, where $M - 1$ is even. By the proof of the even case, we know that each of the $M-1$ games are reduced to a state that is one move away from a terminal position. Assume that b has not been reduced to such a state, but assume b has been moved to some state b'. Because the $M-1$ sub-games give each player an equal amount of points, the winner is determined by the position of b'. If b' is an N-Position, then G is an N-Position; if b' is a P-Position, then G is a P-Position. Thus, the winner of G is determined by b.

□

6 Conclusion and Remarks

After analyzing many specific, small cases of Combined Games, as well as two broader patterns observed in the Normal and Normal and the M Meiseré Combined Games, we draw many conclusions relating to these games. One observation is that in many cases there is no winning strategy; rather, it is optimal for a Combined Game to end in a draw, as we have seen in many of our analyzed games. We also observe that both the Tic-Tac-Toe Effect and the Skip-and-Lose Principle are crucial to analyzing and finding the winning or optimal strategy. And though not specified in previous sections, Adaptive Learning is used to confirm many of our theorems, and we conclude that Adaptive Learning must consider drawing strategies in addition to winning strategies. Our final observation is that the search space of Combined Games, even one with just two sub-games, is enormous and much bigger than any non-combined game.

7 Future Directives

We have given a brief introduction to the field of Combined Games, and thus much work remains. Future analysis of Combined Games is expected to yield general patterns about N Normal Games being played simultaneously, as well as a general conjecture about any amount of games being played in combination. Several classes of games remain unexplored in combination, which include pre-determined games, non pre-determined games, and games where players have different pieces. One such area of future research is examining what happens when playing a pre-determined game in combination with another game; does the game stay pre-determined? Finally, the Combinatorics of Combined Games remains largely open and difficult to analyze, considering the sheer size of the Combined Games space.

References

[1] Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy. *Winning Ways for your Mathematical Plays*, volume One. A K Peters, Ltd., second edition, 2001.

[2] Robert A. Hearn and Erik D. Demaine. *Games, Puzzles, and Computation*. A K Peters, Ltd., 2009.