Algebra Qualifying Exam: Fall 2017
September 20, 2017

Instructions: JUSTIFY YOUR ANSWERS. LABEL YOUR ANSWERS CLEARLY. Do as many problems as you can, as completely as you can. The exam is two and one-half hours. No notes, books, or calculators. As usual, $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote the ring of integers, the field of rational numbers, the field of real numbers and the field of complex numbers respectively, $\mathbb{F}_q$ denotes a finite field with q elements, $M_n(R)$ denotes the ring of $n \times n$ matrices over a ring R, and $\mathrm{GL}_n(R)$ denotes the group of invertible matrices in $M_n(R)$.

1. **(10 points)** Suppose $A \in M_3(\mathbb{C})$ has eigenvalues -1 and 2 (and no other eigenvalues). Let $c_A(x) \in \mathbb{C}[x]$ denote the characteristic polynomial of A, and $m_A(x) \in \mathbb{C}[x]$ the minimal polynomial. Which pairs $(c_A(x), m_A(x))$ can occur? For each pair that can occur, give an explicit example of a matrix A with those characteristic and minimal polynomials.

2. **(10 points)** Prove that no group of order 150 is simple.

3. **(10 points)** Suppose G is an abelian group, and H_1, H_2 are subgroups. Either prove the following statement, or find a counterexample:

   If $G/H_1 \cong G/H_2$, *then* $H_1 \cong H_2$.

4. **(10 points)** Determine up to isomorphism all $\mathbb{F}_2[x]$-modules of order 4.

5. **(10 points)** Suppose that K is a field of characteristic 0, and L is the splitting field of the irreducible polynomial $f(x) \in K[x]$. Prove that if $\mathrm{Gal}(L/K)$ is abelian, and if $a \in L$ is a root of f, then $L = K(a)$.

6. **(10 points)**

   (a) Let $f(x) = x^{31} - 1 \in \mathbb{F}_2[x]$. What is $\mathrm{Gal}(f)$?

   (b) Let $f(x) = x^{31} - 1 \in \mathbb{F}_5[x]$. What is $\mathrm{Gal}(f)$?

7. **(10 points)** Which of the following ideals of $\mathbb{Z}[x, y]$ are prime? Which are maximal? Justify your answer.

   $$(x, y), \quad (x, 3y), \quad (x^2 + 1, y), \quad (x^2 + 1, 3, y), \quad (x^2 + 1, 5, y).$$

8. **(10 points)** If N is a finite normal subgroup of a group G, $d \in \mathbb{Z}_{>0}$, and G/N has an element of order d, then so does G.

9. **(20 points)** Indicate whether each of the following statements is True or False, and give a brief justification.

   (a) Every commutative ring with identity, with exactly 200 elements, has zero divisors.

   (b) For every prime p there is a nonzero homomorphism from $\mathbb{Z}[i]$ to $\mathbb{F}_p$.

   (c) The center of a non-abelian group G is always properly contained in some abelian subgroup.

   (d) If K is a subfield of F and F is isomorphic to K as fields, then $F = K$.

   (e) For every integral domain R and every R-module M, the set of torsion elements is a submodule. (We say $m \in M$ is a torsion element if there is a nonzero $r \in R$ such that $rm = 0$.)