Instructions: JUSTIFY YOUR ANSWERS. LABEL YOUR ANSWERS CLEARLY. Each of the 10 questions is worth 10 points. Do as many problems as you can, as completely as you can. The exam is two and one-half hours. No notes, books, or calculators. As usual, $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote the ring of integers, the field of rational numbers, the field of real numbers and the field of complex numbers respectively, $\mathbb{F}_q$ denotes a finite field with q elements.

1. Classify groups of order $2018 = 2 * 1009$? Justify your answer. You can assume that 1009 is a prime number.

2. Let P be a group of order $|P| = p^r$ for some prime p.

(1) Prove that $Z(P) \neq 1$.
(2) Prove that P is solvable.

3. Let $\mathfrak{m} \subset R = \mathbb{Z}[x_1, \ldots, x_n]$ be a maximal ideal. Prove that $R/\mathfrak{m}$ is a finite field.

4. Let R be a UFD and assume that any ideal in I is finitely generated. Suppose that for every nonzero a, b in R, and any $d = gcd(a, b)$ in R is expressible as $d = ra + sb$ for some $r, s \in R$. Prove that R is a PID.

5. Classify all finite abelian groups G such that $G \otimes_{\mathbb{Z}} (\mathbb{Z}/9\mathbb{Z}) \simeq G$.

6. Let F be a field and let A and B be non-singular 3×3 matrices over F. Suppose that $B^{-1}AB = 2A$.

(1) find the characteristic of F;
(2) if n is a positive or negative integer not divisible by 3, prove that the matrix A^n has trace 0;
(3) prove that the characteristic polynomial of A is $X^3 - a$ for some $a \in F$.

7. Let K be a field, and let A be an $n \times n$-matrix over K. Suppose that $f \in K[x]$ is an irreducible polynomial such that $f(A) = 0$. Show that $\deg(f)|n$.

8. Let F be a field and let $f(x) \in F[x]$ be an irreducible polynomial. Suppose E is a splitting field for $f(x)$ over F and assume that there exists an element $\alpha \in E$ such that both α and $\alpha + 1$ are roots of $f(x)$.

(1) show that the characteristic of F is not zero;
(2) prove that there exists a field L between F and E such that the degree $[E : L]$ is equal to the characteristic of F.

9. Let $\mathbb{F}_q$ be a finite field and let $\alpha \in \mathbb{F}_q^*$. Let K be a splitting field over $\mathbb{F}_q$ of $X^{q+1} - \alpha$. Prove that $[K : \mathbb{F}_q] = 2$.

10. For the alternating group A_4,

(1) Classify the conjugacy classes of A_4.
(2) Construct the character table of A_4.