ALGEBRA COMPREHENSIVE EXAM

Thursday, 18 June 2020

Math Exam ID#: ____________________

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total		100

Instructions: Each problem is worth 10 points. Unless otherwise specified, all rings are commutative, have unity, are not the zero ring, and all ring homomorphisms map unity to unity. Write your proofs clearly using complete sentences. Your proofs will be graded based on clarity as well as correctness; a correct answer will not receive full credit if the reasoning is difficult to follow. Good luck.

1. Assume G is a group, and assume $H_0 \leq G$ is a subgroup with the property that, for every non-trivial subgroup $H \leq G$, we have $H_0 \leq H$. Prove that H_0 is in the center of G.

2. How many elements of order 7 does a simple group of order 168 have? Prove your answer. (Note. $168 = 2^3 \cdot 3 \cdot 7$)

3. Let X denote a set on which the group A_4 acts transitively. What are the possibilities for the cardinality of X? Prove your answer.

4. Let $S = \{\sum a_i x^i \in \mathbb{R}[x] \colon a_1 = 0\}$. You may use without proof that S is a subring of $\mathbb{R}[x]$.
   a. Give an example of an ideal in S which is not principal. Briefly explain.
   b. Is S a UFD? Briefly explain.

5. All of the following are isomorphic as $\mathbb{R}$-vector spaces, but only two of the following are isomorphic as rings. Which two? Explain why they are isomorphic as rings.
   i. $\mathbb{C} \times \mathbb{C}$
   ii. $\mathbb{C}[x]/(x^2)$
   iii. $\mathbb{C}[x]/(x^2+1)$
   iv. $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$
   v. $\mathbb{R}[x]/(x^4)$

6. Let $I \subseteq \mathbb{Z}[\sqrt{-3}]$ be a non-zero ideal. Prove that the quotient ring $\mathbb{Z}[\sqrt{-3}]/I$ is finite.

7. Give an example of two invertible matrices $A, B \in GL_4(\mathbb{Q})$ such that A and B have the same minimal polynomial and the same characteristic polynomial, but are not similar matrices. Briefly explain your answer. (Notice the requirement that A, B be invertible.)

8. Let F be a field and let $A, B \in M_{n\times n}(F)$.
   a. Prove that 0 is an eigenvalue of AB if and only if 0 is an eigenvalue of BA.
   b. Prove that $\lambda \in F$ is an eigenvalue of AB if and only if λ is an eigenvalue of BA.

9. Let E/F be a field extension of degree 12. Prove that there exist $\alpha_1, \alpha_2, \alpha_3 \in E$ such that $E = F(\alpha_1, \alpha_2, \alpha_3)$.

10. Let k be a finite field and let the function $\sigma : k \to k$ be given by $x \mapsto x^2$.
    a. Assume the cardinality of k is even. Is σ surjective? Briefly explain.
    b. Assume the cardinality of k is odd. Is σ surjective? Briefly explain.