ALGEBRA COMPREHENSIVE EXAM

Monday, 14 September 2020

Math Exam ID#: ____________________

Instructions: The true-false problem is worth 20 points; all other problems are worth 10 points. Unless otherwise specified, all rings are commutative, have unity, are not the zero ring, and all ring homomorphisms map unity to unity. Write your proofs clearly using complete sentences. Your proofs will be graded based on clarity as well as correctness; a correct answer will not receive full credit if the reasoning is difficult to follow. Good luck.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		20
Total		100

1. For each of the following finite groups, give an explicit product of cyclic groups of the form $(\mathbb{Z}/n_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_k\mathbb{Z})$ to which it is isomorphic. No explanation is necessary.
   a. $Z(D_{32})$, the center of the dihedral group of order 64.
   b. $Z(A_4)$, the center of A_4.
   c. $(\mathbb{F}_{64}, +)$, the underlying additive group of the field $\mathbb{F}_{64}$.
   d. $(\mathbb{F}_{64}^{\times}, \cdot)$, the unit group of the field $\mathbb{F}_{64}$.
   e. $\{z \in \mathbb{C} \colon z^{64} = 1\}$, with group operation multiplication.
2. Let G be a finite index subgroup of $\mathbb{R}$. Prove that $G = \mathbb{R}$.
3. Assume G is a group and $H \leq G$ is a subgroup of finite index n. Prove that G contains a normal subgroup N such that $N \subseteq H$ and such that $[G : N] \leq n!$. (Hint. Consider a suitable group action.)
4. a. Give two examples of infinite rings R with the following property: the ring R is not a field, and for every non-zero ideal $I \subseteq R$, the quotient ring R/I is finite. No explanation is necessary.
   b. Prove that in any such ring, it is impossible to have a chain
   $$\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \mathfrak{p}_3$$
   of three distinct prime ideals.
5. Let A be an $n \times n$ matrix over a field F, and let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in F[x]$ be the characteristic polynomial of A. Prove that the trace of A is equal to $-a_{n-1}$.
6. Does there exist a matrix $A \in M_{5\times 5}(\mathbb{R})$ satisfying $A^2 + 2A = -2I_5$? Prove your answer. (Here I_5 denotes the 5×5 identity matrix.)
7. Assume $f(x) \in \mathbb{Q}[x]$ is an irreducible degree 5 polynomial, and assume there exists $\alpha \in \mathbb{C}$ such that both α and $-\alpha$ are roots of $f(x)$. Prove that no splitting field of $f(x)$ over $\mathbb{Q}$ can contain a primitive 5-th root of unity.
8. Is there a scalar multiplication $\mathbb{Z}[\sqrt{2}] \times \mathbb{Z}/5\mathbb{Z} \to \mathbb{Z}/5\mathbb{Z}$ making the abelian group $\mathbb{Z}/5\mathbb{Z}$ into a $\mathbb{Z}[\sqrt{2}]$-module? Prove your answer.
9. Answer True or False to each of the following statements and briefly explain your answers.
   a. True or false: If F is a finite field and $A \in M_{n\times n}(F)$ is an invertible matrix, then there exists a positive integer m such that $A^m = I_n$.
   b. True or false: If R is an integral domain and $I \cap J = \{0\}$, where I and J are ideals in R, then $I = \{0\}$ or $J = \{0\}$.
   c. True or false: If p is prime and $H \leq S_p$ is a subgroup such that the order of H is divisible by p, then H contains a p-cycle.
   d. True or false: If $\varphi : K \to L$ is a ring homomorphism, where K and L are fields, then K and L must have the same characteristic. (By our definition, ring homomorphisms send unity to unity.)