# Analysis Comprehensive Exam

June 19, 2020 (Friday)

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**SCORES:**

**1.** ____________ /10

**2.** ____________ /10

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**Problem 1:** Let $E \subset \mathbb{R}$ be uncountable. Prove that $E$ has uncountably many limit points.

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**Problem 2:** Suppose that $X$ is a metric space and $K \subseteq X$ is compact. Prove that there are $a, b \in K$ that are "as far apart as possible," that is, $d(x, y) \leq d(a, b)$ for all $x, y \in K$.

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**Problem 3:** Let $I(x) = 1$ if $x \geq 0$ and $I(x) = 0$ if $x < 0$. Let

$$\alpha(x) := \sum_{n=1}^{\infty} 2^{-n} I(x - 2^{-n}).$$

Compute the Riemann–Stieltjes integral $\int_0^1 x\, d\alpha$.

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**Problem 4:** Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$. Prove that there is $\theta \in (a, b)$ such that

$$(f(b) - f(a))g'(\theta) = (g(b) - g(a))f'(\theta).$$

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**Problem 5:** Let $\{f_n\}$ be a bounded sequence of convex functions on $[-2, 2]$. Show that there is a subsequence $\{f_{n_k}\}$ that converges uniformly on $[-1, 1]$.

(Recall that a function $f$ is convex on an interval $I$ if for all $0 \leq \lambda \leq 1$ and all $x, y \in I$, one has $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$.)

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**Problem 6:** Let $X$ denote the set of non-decreasing functions $f : [0,1] \to [0,1]$. By *non-decreasing*, we mean that $x \leq y$ implies that $f(x) \leq f(y)$. We equip $X$ with the metric: $d(f,g) = \sup\{|f(x) - g(x)|; x \in [0,1]\}$.

a). Prove that $(X,d)$ is complete.

b). Prove that $(X,d)$ is not compact.

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**Problem 7:** Let $B_R(\mathbf{0}) = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_1^2 + x_2^2 + \cdots + x_n^2 \leq R^2\}$ be the ball of radius $R$ in the $n$-dimensional Euclidean space. Compute the volume of $B_R(\mathbf{0})$.

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**Problem 8:** For any bounded real-valued function $u \in \mathcal{R}([a,b])$ (Riemann integrable), we define the so-called $L^2$-*norm* of $u$:

$$||u||_2 = \left( \int_a^b |u(x)|^2 dx \right)^{1/2}.$$

Prove that

a). If $f, g \in \mathcal{R}([a,b])$, then

$$\left| \int_a^b f \cdot g dx \right| \leq ||f||_2 \cdot ||g||_2.$$

b). If $f, g, h \in \mathcal{R}([a,b])$, then

$$||f - h||_2 \leq ||f - g||_2 + ||g - h||_2.$$