Print Your Examination number:

Complex Qualifying Examination

September 15, 2021

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Notation: Let $D(z_0, r)$ be the disk centered at z_0 and radius r in the complex plane $\mathbf{C}$, let $U = \{z \in \mathbf{C} : \text{Im z} > 0\}$ be the upper half plane.

1. Compute the integral

$$\int_{-\pi}^{\pi} \frac{1}{5 + 3\cos\theta} d\theta$$

2. Let $p(z)$ be a polynomial of degree d. Prove that if

$$\{z \in \mathbf{C} : p(z) = 0\} \subset D(0,1)$$

then

$$\{z \in \mathbf{C} : p'(z) = 0\} \subset D(0,1).$$

3. Let $f(z) = \sum_{j=0}^{\infty} c_j z^j$ be entire holomorphic and satisfy

$$|f(z)| \leq e^{|z|}, \quad z \in \mathbf{C}.$$

Prove

$$|c_n| \leq \left(\frac{e}{n}\right)^n, \quad n = 1, 2, 3, \cdots.$$

4. Let $f(z) : D(0,1) \to D(0,1)$ be holomorphic with $f(0) = 3^{-10} e^{\frac{i\pi}{10}}$.

(i) Prove that f has at most 10 zeros in $\overline{D(0,1/3)}$ counting multiplicities;

(ii) Provide an example of such f which has 10 zeros in $\overline{D(0,1/3)}$ counting multiplicities.

5. Prove or disprove there is a holomorphic function f in $D = \mathbf{C} \setminus D(0,4)$ such that

$$f'(z) = \frac{\cos(\pi z)}{z-1} + \frac{z}{(z-2)} + \frac{e^{3-z}}{(z-3)^2}.$$

6. Find a conformal map which maps $D = \{z = x + iy : 0 < x < 1, y > 0\}$ onto the unit disc $D(0, 1)$.

7. Prove that all zeros for

$$f(z) = z^8 + 4z^3 - 5$$

lie in the annulus $D(0,2) \setminus D(0,1)$.

8. Find all entire holomorphic functions f satisfying

$$\left|f\Big(\frac{1}{\log(n+1)}\Big)\right| \leq \frac{1}{n}, \qquad \text{for all positive integers } n.$$

9. Let f be holomorphic in the unit disc $D(0,1)$ satisfying $|f(z)| \leq 1$ on $D(0,1)$. Prove

$$\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)||z|}, \quad z \in D(0,1).$$