Print Your Examination number:

Complex Qualifying Examination
June, 2021

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Notation: Let $D(z_0, r)$ be the disk centered at $z_0$ and radius $r$ in the complex plane $\mathbb{C}$, let $U = \{z \in \mathbb{C} : \text{Im } z > 0\}$ be the upper half plane.

1. Show that for any integer $n \geq 2$
\[
\int_0^\infty \frac{1}{1 + x^n} dx = \frac{\pi}{\sin \left(\frac{\pi}{n}\right)}.
\]

2. Let $f$ be holomorphic in $D(0, 2)$ and continuous on $\overline{D(0, 2)}$. If $|f(z)|$ is a constant on the boundary of $D(0, 2)$ then $f$ is either a constant or has finite many zeros in $D(0, 2)$.

3. Define the Bernoulli numbers $B_n$ by the power series
\[
\frac{z}{e^z - 1} = \sum_{n=0}^\infty \frac{B_n}{n!} z^n.
\]

(i) Prove that $B_0 = 1$ and
\[
\sum_{k=0}^{n-1} \frac{B_k}{(n-k)!} = 0, \quad \text{for any } n > 1.
\]

(ii) Find $B_1, B_2, B_3$ and $B_4$.

4. Let
\[
F(z) = \sum_{n=0}^\infty \frac{1}{2^n} z^{2^n}.
\]
Prove the following two statements:
   (i) $F$ is holomorphic in $D(0, 1)$ and continuous on $\overline{D(0, 1)}$;
   (ii) Every point of $\partial D(0, 1)$ is singular point for $f$.

5. Let $D$ be a bounded domain in $\mathbb{C}$ with $C^1$ boundary and $0 \in D$. Prove or disprove there are a sequence polynomials $\{p_n\}_{n=1}^\infty$ such that $p_n(z) \to \frac{1}{z^2}$ as $n \to \infty$ uniformly on $\partial D$.

6. Let $u$ be harmonic in $D(0, 1) \setminus \{0\}$ satisfying
\[
\lim_{z \to 0} \frac{u(z)}{\log |z|} = 0.
\]
Prove that $u$ is harmonic in $D(0, 1)$.  

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7. Prove that

\[(3 - e)|z| < |e^z - 1| < (e - 1)|z|, \quad \text{for any } z \in D(0, 1) \setminus \{0\}.
\]

8. Suppose that \(f\) is holomorphic in \(D(0, 1) \setminus \{0\}\) such that

\[f\left(\frac{1}{n}\right) = \frac{(-1)^n}{n}, \quad \text{for all positive integers } n.
\]

Prove that \(\lim_{n \to \infty} |f(z)|\) does not exist.

9. Prove that all zeros of the polynomial

\[p_n(z) = z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n\]

lie in the disk \(D(0, R)\), where

\[R = 1 + \max\{|a_j| : 1 \leq j \leq n}\].