Print Your Examination number:

Complex Qualifying Examination

June, 2021

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Notation: Let $D(z_0, r)$ be the disk centered at z_0 and radius r in the complex plane $\mathbf{C}$, let $U = \{z \in \mathbf{C} : \mathbf{Im\ z} > \mathbf{0}\}$ be the upper half plane.

1. Show that for any integer $n \geq 2$

$$\int_0^\infty \frac{1}{1+x^n} dx = \frac{\pi}{\sin(\frac{\pi}{n})}.$$

2. Let f be holomorphic in $D(0,2)$ and continuous on $\overline{D(0,2)}$. If $|f(z)|$ is a constant on the boundary of $D(0,2)$ then f is either a constant or has finite many zeros in $D(0,2)$.

3. Define the Bernoulli numbers B_n by the power series

$$\frac{z}{e^z - 1} = \sum_{n=0}^\infty \frac{B_n}{n!} z^n$$

(i) Prove that $B_0 = 1$ and

$$\sum_{k=0}^{n-1} \frac{B_k}{(n-k)!k!} = 0, \qquad \text{for any } n > 1.$$

(ii) Find B_1, B_2, B_3 and B_4.

4. Let

$$F(z) = \sum_{n=0}^\infty \frac{1}{2^n} z^{2^n}.$$

Prove the following two statements:

(i) F is holomorphic in $D(0,1)$ and continuous on $\overline{D(0,1)}$;

(ii) Every point of $\partial D(0,1)$ is singular point for f.

5. Let D be a bounded domain in $\mathbf{C}$ with C^1 boundary and $0 \in D$. Prove or disprove there are a sequence polynomials $\{p_n\}_{n=1}^\infty$ such that $p_n(z) \to \frac{1}{z^2}$ as $n \to \infty$ uniformly on ∂D.

6. Let u be harmonic in $D(0,1) \setminus \{0\}$ satisfying

$$\lim_{z \to 0} \frac{u(z)}{\log |z|} = 0.$$

Prove that u is harmonic in $D(0,1)$.

7. Prove that

$$(3-e)|z| < |e^z - 1| < (e-1)|z|, \quad \text{for any } z \in D(0,1) \setminus \{0\}.$$

8. Suppose that f is holomorphic in $D(0,1) \setminus \{0\}$ such that

$$f(\frac{1}{n}) = \frac{(-1)^n}{n}, \qquad \text{for all positive integers } n.$$

Prove that $\lim_{n\to\infty} |f(z)|$ does not exist.

9. Prove that all zeros of the polynomial

$$p_n(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

lie in the disk $D(0, R)$, where

$$R = 1 + \max\{|a_j| : 1 \le j \le n\}.$$