ALGEBRA QUALIFYING EXAM
JUNE 17, 2021

Math Exam ID#: ____________________

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total		100

Instructions: There are ten problems and each problem is worth 10 points. Unless otherwise specified, all rings are commutative, have unity, are not the zero ring, and all ring homomorphisms map unity to unity. Write your proofs clearly using complete sentences. Your proofs will be graded based on clarity as well as correctness; a correct answer will not receive full credit if the reasoning is difficult to follow. Good luck.

1. a. Let G be a group of order n. Prove that G is isomorphic to a subgroup of S_n.

   b. Is every group of order 6 isomorphic to a subgroup of A_6? Explain.

2. Let G denote a finite group with identity element e. Assume $n > 1$ is a fixed integer for which $x^n y^n = (xy)^n$ for all $x, y \in G$.

   a. Prove that such an $n > 1$ always exists.

   b. Define
   $$G^{(n)} = \{x^n \mid x \in G\} \qquad \text{and} \qquad G_{(n)} = \{x \in G \mid x^n = e\}.$$
   Prove that the index of $G_{(n)}$ in G is equal to the order of $G^{(n)}$.

3. a. Define what it means for an integral domain R to be a Euclidean domain.

   b. Prove that $\mathbb{Z}[i]$ is a Euclidean domain.

4. Does there exist a ring R (commutative, with unity) such that its underlying additive group $(R, +)$ is isomorphic to $\mathbb{Q}/\mathbb{Z}$? Prove your answer.

5. Let R be a commutative ring.

   a. Assume that every ideal of R is finitely generated. Prove that, for every countably infinite chain $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ of ideals in R, there exists an integer $N > 0$ such that $I_n = I_{n+1}$ for all $n \geq N$.

   b. Give an example of a chain of ideals in a PID R which contains 5 distinct ideals:
   $$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq I_4 \subsetneq I_5$$

6. Assume $\alpha \in \mathbb{C}$ is algebraic over $\mathbb{Q}$ and assume $\alpha \neq 0$. Prove there exists a polynomial $f(x) \in \mathbb{Q}[x]$ such that $f(\alpha) = \alpha^{-1}$.

7. Let K denote a splitting field of $(x^2 - 2)(x^2 + 3)$ over $\mathbb{Q}$.

   a. Determine the degree of $K/\mathbb{Q}$. Briefly justify your answer.

   b. Write out the elements in $\mathrm{Gal}(K/\mathbb{Q})$. (To receive full credit, explain how you know that the elements you write down are actually automorphisms.)

   c. Write out the intermediate fields between K and $\mathbb{Q}$, and for each intermediate field, list the subgroup of $\mathrm{Gal}(K/\mathbb{Q})$ to which it corresponds. (No justification is necessary for this part.)

8. Let p be a prime and m be a positive integer satisfying $1 < m < p$. Let K/F be a Galois extension where $[K\colon F] = mp^n$ where n is a positive integer. Prove that there is a subfield E of K containing F with $F \subsetneq E \subsetneq K$ such that E/F is Galois.

9. Determine the number of isomorphism classes of $\mathbb{F}_3[x]$-modules of order 9. Include a full justification of your claim.

10. Let m, n be positive integers and let $d = \gcd(m, n)$. Prove that there is an isomorphism of $\mathbb{Z}$-modules (equivalently, of abelian groups),

$$(\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}.$$