ALGEBRA QUALIFYING EXAM
SEPTEMBER 2021

Math Exam ID#: ____________________

Instructions: The last question is worth 20 points; all other questions are worth 10 points each. Unless otherwise specified, all rings are commutative, have unity, are not the zero ring, and all ring homomorphisms map unity to unity. Write your proofs clearly using complete sentences. Your proofs will be graded based on clarity as well as correctness; a correct answer will not receive full credit if the reasoning is difficult to follow. Good luck.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		20
Total		100

1. Let V be a vector space over a field K. Let $T : V \to V$ be a linear transformation. Let $v, w \in V$ denote eigenvectors of T with distinct eigenvalues. Prove that the set $\{v, w\}$ is linearly independent.

2. a. Let p denote an odd prime. Prove that, up to isomorphism, there are precisely two groups of order $2p$. (The same holds if $p = 2$, but you don't need to show that.)

   b. Let n be a positive integer and p be a prime. What is the order of a Sylow p-subgroup of $GL_n(\mathbb{Z}/p\mathbb{Z})$? Give a brief explanation.

3. Assume G is a group and A, B are normal subgroups of G such that the groups G/A and G/B are abelian. You may use without explanation that $A \cap B$ is a normal subgroup of G. Prove that $G/(A \cap B)$ is also abelian.

4. a. Let p denote a prime and let $H \leq S_p$ denote a subgroup which acts transitively on $\{1, 2, \ldots, p\}$. Show that H contains a p-cycle.

   b. Give an example of a subgroup of S_4 which acts transitively on $\{1, 2, 3, 4\}$ and which does not contain a 4-cycle.

5. Let $f(x) \in \mathbb{Z}[x]$ be any element. Prove that the principal ideal $(f(x))$ is not a maximal ideal in $\mathbb{Z}[x]$.

6. Let F be a field with 7^3 elements. Prove that 3 is not a square of an element in F.

7. Let p be a prime and $\zeta_p = e^{2\pi i/p}$.

   a. For which p does there exist a subfield E of $\mathbb{Q}(\zeta_p)$ such that $E/\mathbb{Q}$ is a Galois extension with $\mathrm{Gal}(E/\mathbb{Q}) \cong \mathbb{Z}/3\mathbb{Z}$. Justify your answer.

   b. Let $E/\mathbb{Q}$ be an extension as in part (a). Prove that $E \subset \mathbb{R}$.

8. Let R be an integral domain and let M be an R-module. Recall that $Tor(M)$ denotes all $m \in M$ such that there exists $r \in R \setminus \{0\}$ such that $rm = 0$. An R-module M is called *torsion-free* if $Tor(M) = \{0\}$.

   a. Prove that $Tor(M)$ is an R-submodule of M.

   b. Prove that $M/Tor(M)$ is torsion-free.

9. (20 points) For each of the following, either give an example or briefly explain why no such example exists.

   a. A group G and a subgroup $H \leq G$ such that G is simple and H is not simple.

   b. A non-abelian group G such that the map $G \to G$ given by $g \mapsto g^2$ is a group homomorphism.

   c. A 3×3 real matrix M such that $M^3 = -I_3$, where I_3 is the 3×3 identity matrix.

   d. A field K and a polynomial $f(x) \in K[x]$ such that the splitting field of $f(x)$ over K is not a Galois extension of K.