Print Your Examination number:

Complex Qualifying Examination

January 4, 2022

Table of your scores

Problem 1	______ / 10
Problem 2	______ / 10
Problem 3	______ / 10
Problem 4	______ / 10
Problem 5	______ / 10
Problem 6	______ / 10
Problem 7	______ / 10
Problem 8	______ / 10
Problem 9	______ / 10

Total ______ / 90

1. Let $P(z) = z^n + a_1 z^{n-1} + \ldots + a_n$. Show that either $P(z) = z^n$ identically, or there exists a point $z_0 \in \mathbf{C}$, $|z_0| = 1$, such that $|P(z_0)| > 1$.
2. Find the number of roots of the equation $z^4 + z^3 - 4z + 1 = 0$ in the domain $\{z \in \mathbf{C} : 1 < |z| < 2\}$. Roots are counted as many times as its multiplicity.
3. Find a conformal map φ, mapping the domain $\{z \in \mathbf{C} : |\arg(\mathrm{z})| < \pi/4\}$ onto the unit disk $\{z \in \mathbf{C} : |z| < 1\}$ with properties $\varphi(1) = 0$ and $\arg \varphi'(1) = \pi$.
4. Evaluate the following integral

$$\int_{\partial D(0,2)} \frac{1}{z^n(1-z)} dz \quad \text{for all } n = 0, 1, 2, \cdots,$$

where $D(0,2) \subset \mathbf{C}$ is the disc centered at 0 with radius 2.
5. (a) Classify the singularities of the function

$$f(z) = \frac{z}{\sin z}$$

Include the point at infinity.
(b) Find a Laurent expansion, valid in the region $|z+1| > 3$, for

$$f(z) = \frac{7z - 2}{z^3 - z^2 - 2z}.$$

Find the residue of f at $z = 0$.
6. Let f be holomorphic in $\mathbf{C}$ with

$$|f(z)| \geq |z|^5, \quad \text{for } |z| > 1$$

Prove or disprove that f is a polynomial.
7. (a) Prove that all points on $\partial D(0,1)$ are singular points of the function

$$f(z) = \sum_{n=1}^{\infty} \frac{n}{4^n} z^{2^n}. \tag{1}$$

(b) Prove that f defined by (1) is differentiable function on $\overline{D}(0,1)$.

8. Let f be holomorphic in $D(0,1)$ with $f(0) = f'(0) = 0$. Prove that the series

$$\sum_{n=2}^{\infty} f\Big(\frac{1}{\sqrt{n}\log(n+1)}\Big)$$

converges.

9. Let $f : D(0,1) \setminus \{0\}$ be a holomorphic with

$$\int_{D(0,1)} |f(z)|^{3/2} dA(z) = 1.$$

Prove that $z = 0$ is a pole of order 1 or a removable singularity of the function f.