ALGEBRA QUALIFYING EXAM
JANUARY 2022

Math Exam ID#: ____________________

Instructions: The last question is worth 20 points; all other questions are worth 10 points each. Unless otherwise specified, all rings are commutative, have unity, are not the zero ring, and all ring homomorphisms map unity to unity. Write your proofs clearly using complete sentences. Your proofs will be graded based on clarity as well as correctness; a correct answer will not receive full credit if the reasoning is difficult to follow. Good luck.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		20
Total		100

1. Assume H is a normal subgroup of a group G. Let S denote the set of left cosets of H in G. Prove that the following operation is well-defined:
$$S \times S \to S, \qquad (aH, bH) \mapsto abH.$$

2. Let $n \geq 5$ be an integer. You may use without proof that A_n is the only non-trivial proper normal subgroup of S_n. Let H be a subgroup of S_n such that $1 < [S_n\colon H] < n$. Prove that $H = A_n$.

3. Prove that if G is a group of order 12, then G is isomorphic to a semidirect product $H \rtimes_\varphi K$ where $H, K \leq G$ are proper non-trivial subgroups of G.

4. Assume M is an $n \times n$ matrix over some field F, and assume $\vec{v} \in F^n$ is a vector such that $(M - \lambda I)\, \vec{v} \neq \vec{0}$ and $(M - \lambda I)^2\, \vec{v} = \vec{0}$ for some $\lambda \in F$, where I denotes the $n \times n$ identity matrix. Prove that M is not diagonalizable.

5. Let R be a ring and assume I is a prime ideal in R. Assume R/I satisfies the *descending chain condition*, meaning that for every countably infinite descending chain of ideals $J_1 \supseteq J_2 \supseteq \cdots$ in R/I, there exists $N > 0$ such that $J_n = J_{n+1}$ for all integers $n \geq N$. Prove that I is a maximal ideal in R.

6. Consider the subring S of $\mathbb{Q}[x]$ defined as follows: $S = \{f(x) \in \mathbb{Q}[x] \mid f(0) \in \mathbb{Z}\}$. Prove that the element $x \in S$ cannot be expressed as a product of irreducible elements.

7. Determine the structure (as a direct product of cyclic groups) of the group of units of the ring $\mathbb{F}_5[x]/(x^3 + 1)$.

8. Let $n \geq 2$ be an integer, and let ζ_n denote a primitive n-th root of unity.
   a. Give an explicit bijection between $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ and $(\mathbb{Z}/n\mathbb{Z})^\times$. You don't have to prove it's a bijection.
   b. What is the degree of $\mathbb{Q}(\zeta_{11} + \zeta_{11}^3 + \zeta_{11}^9 + \zeta_{11}^5 + \zeta_{11}^4)/\mathbb{Q}$? Justify your answer.

9. (20 points) For each of the following, either give an example or briefly explain why no such example exists.
   a. Finite order elements a, b in a group G such that ab has infinite order.
   b. A surjective group homomorphism $\mathbb{Q} \to \mathbb{Z}$.
   c. A 3×3 real matrix M such that $M^2 = -I_3$, where I_3 is the 3×3 identity matrix.
   d. A tower of fields $K \supseteq E \supseteq F$ such that K/E and E/F are Galois extensions but K/F is not a Galois extension.