Print Your Examination number: ____________

Complex Qualifying Examination
1:00pm–3:30pm, September 20, 2023 at RH 306

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Notation. Let $D(a, r)$ denote the disc in the complex plane $\mathbb{C}$ centered at $a$ with radius $r$ and $\partial D(a, r) = \{z \in \mathbb{C} : |z - a| = r\}$. Let $\mathbb{R}$ denote the set of all real numbers. Let $H = \{z \in \mathbb{C} : \text{Im}z > 0\}$ be the upper half plane.

Problem 1: Find the line integral:

$$\int_{\gamma} z \sin z \, dz$$

where $\gamma$ the curve from $-1$ to $1$ taken along a semicircle.
Problem 2: A domain $\Omega$ in $\mathbb{C}$ is said to be holomorphic simply connected if for any holomorphic function $f$ on $\Omega$ and any simple closed piecewise $C^1$ curve $\gamma$ in $\Omega$, one has $\int_{\gamma} f(z)dz = 0$.

(a) Prove that

$$\Omega = \{z = x + iy \in \mathbb{C} : y > x\}$$

is holomorphic simply connected.

(b) Prove that $\Omega = D(0,1) \setminus \{0\}$ is not holomorphic simply connected.
Problem 3:

a) Prove that the series \( \sum_{n=0}^{\infty} e^{n(1+i)}z \) converges to a holomorphic function in a neighborhood of the point \( z_0 = i \).

b) If \( f(z) = \sum_{n=0}^{\infty} e^{n(1+i)}z \) is represented as a power series

\[
f(z) = \sum_{n=0}^{\infty} a_n (z - i)^n,
\]

what is its radius of convergence?
**Problem 4:** Find an explicit conformal mapping of the domain

\[ U = \{ z \in \mathbb{C} : |z - i| < \sqrt{2} \text{ and } |z + i| < \sqrt{2} \} \]

onto the unit disc.
Problem 5: If \( f : \mathbb{C} \to \mathbb{C} \) is an entire function, one can define a sequence of functions \( \{ f^{(n)} \} \) by

\[
f^{(1)} = f, \quad f^{(2)} = f \circ f, \quad f^{(3)} = f \circ f \circ f, \ldots, \quad f^{(n)} = f^{(n-1)} \circ f, \ldots
\]

Prove or disprove: For any entire function \( f : \mathbb{C} \to \mathbb{C} \), the sequence \( \{ f^{(n)} |_{D(0,1)} \} \) of restrictions of \( f^{(n)} \) to the unit disc form a normal family.
Problem 6:

a) Suppose that \( u : \mathbb{C} \to \mathbb{R} \) is a harmonic function such that \( u|_{\mathbb{R}} = 0 \). Does it imply that \( u \equiv 0 \)?

b) Suppose that \( u : \mathbb{C} \to \mathbb{R} \) is a harmonic function such that \( u|_{\partial D(0,1)} = 0 \). Does it imply that \( u \equiv 0 \)?
Problem 7: Let $f$ be entire holomorphic such that

$$|f(z)|^2 = p_n(x, y), \quad z = x + iy,$$

where $p_n(x, y)$ is a polynomial of $x$ and $y$ of degree $n$. Prove that $f$ is a polynomial of $z$. 
Problem 8: Let $f$ be holomorphic in $D = D(0, 1) \setminus \{0\}$ such that

$$\int_0^{2\pi} |f(re^{i\theta})|d\theta \leq 1, \text{ for all } 0 < r < 1.$$

Prove that $z = 0$ is a removable singularity.