ALGEBRA QUALIFYING EXAM
JUNE 2023

Math Exam ID#: ____________________

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total		100

Instructions: All problems are worth 10 points. Unless otherwise specified, all rings are commutative, have unity, are not the zero ring, and all ring homomorphisms map unity to unity. Write your proofs clearly using complete sentences. Your proofs will be graded based on clarity as well as correctness; a correct answer will not receive full credit if the reasoning is difficult to follow. Good luck.

1. Prove that, up to isomorphism, there are two groups of order 14.

2. a. Let $\mathbb{C}^*$ denote the group of non-zero complex numbers. How many group homomorphisms are there $\phi : S_6 \to \mathbb{C}^*$?

   b. How many group homomorphisms are there $\phi : S_6 \to \mathbb{Z}/3\mathbb{Z}$?

3. Let G be a group that contains no index 2 subgroup. Show that every index 3 subgroup in G is normal. (Do not assume that G is a finite group.)

4. Are the following isomorphic as rings? Prove your answer.
$$\mathbb{R}[x]/(x^3) \text{ and } \mathbb{R}[x]/(x^3 - 8)$$

5. Recall that a non-unit, non-zero element π in a ring R is called *prime* if $\pi \mid rs$ implies $\pi \mid r$ or $\pi \mid s$ (where $r, s \in R$). Assume R is a PID and $\pi_1 \neq \pi_2$ are two distinct prime elements in R and that π_1 is not a unit multiple of π_2 (i.e., that π_1 and π_2 are not associates). Prove there exist $a, b \in R$ such that $a\pi_1 + b\pi_2 = 1$.

6. Let $A \in M_{7\times 7}(\mathbb{C})$ denote a matrix satisfying $A^2 = 0$. What is the largest possible rank that A can have? Justify your answer.

7. Consider the rational numbers $\mathbb{Q}$ as a $\mathbb{Z}$-module. Is $\mathbb{Q}$ a finitely generated $\mathbb{Z}$-module? Prove your answer.

8. Find the Galois group of $f(x) = (x^3 - 2)(x^2 + 3)$ over $\mathbb{Q}$. Justify your answer.

9. a. Let $K \subseteq \mathbb{C}$ and assume K is a (finite) Galois extension of $\mathbb{Q}$. Let $\tau : \mathbb{C} \to \mathbb{C}$ denote complex conjugation. Prove that $\tau(K) = K$.

   b. Assume further that $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$. Prove that $i \notin K$.
   (Possible approach: Assume towards contradiction that $i \in K$ and consider the fixed field of complex conjugation.)

10. For each of the following, either give an example or state that no such example exists. Briefly explain your answers.

    a. A non-trivial finite abelian group A such that $A \otimes_{\mathbb{Z}} A$ is trivial.

    b. Two finite fields which are isomorphic as groups (i.e., whose underlying additive groups are isomorphic) but which are not isomorphic as rings.