ALGEBRA QUALIFYING EXAM
SEPTEMBER 2024

Math Exam ID#: ____________________

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total		100

Instructions: All problems are worth 10 points. Unless otherwise specified, all rings are commutative, have unity, are not the zero ring, and all ring homomorphisms map unity to unity. Write your proofs clearly using complete sentences. Your proofs will be graded based on clarity as well as correctness; a correct answer will not receive full credit if the reasoning is difficult to follow. Good luck.

1. Let G be a group and consider the subgroup $H \leq G \times G$ defined by
$$H := \{(g, g) \colon g \in G\}.$$
You may use without proof that H is indeed a subgroup of $G \times G$. Prove that H is a normal subgroup of $G \times G$ if and only if G is an abelian group.

2. Let p be a prime and let $k \geq 2$ be an integer. Prove that no group of order $p^k(p+1)$ is simple.

3. Let R be a commutative ring with identity. Assume $R[x]$ is a PID. Prove that R is a field.

4. a. Let R be a commutative ring with identity and let $a, b \in R$. Prove that the rings $R/(a, b)$ and $(R/(a))/(\bar{b})$ are isomorphic, where $(\bar{b})$ denotes the principal ideal in $R/(a)$ generated by $\bar{b} := b + (a)$.

   b. Prove that $(5, x^2 + 1)$ is not a maximal ideal in $\mathbb{Z}[x]$.

5. Let R be a commutative ring with unity. We say that R satisfies the *ascending chain condition* (ACC) for ideals if every ascending chain of ideals
$$I_1 \subseteq I_2 \subseteq \cdots$$
eventually stabilizes, i.e., for every such chain, there exists an integer $N > 0$ such that $I_n = I_N$ for all $n \geq N$. Prove that R satisfies the ACC for ideals if and only if every ideal in R is finitely generated.

6. Let R be an integral domain, let M be an R-module, and let N be an R-submodule of M. Assume that N and M/N are both torsion-free. Prove that M is also torsion-free. (Recall that we say an R-module M is *torsion-free* if $rm = 0$ for $r \in R$, $m \in M$ implies $r = 0$ or $m = 0$.)

7. a. Let M be an $n \times n$ complex matrix whose characteristic polynomial is separable (meaning the polynomial has no repeated roots). Assume further that all roots of the characteristic polynomial of M are roots of unity. Prove that there exists a positive integer k such that $M^k = I_n$, where I_n denotes the $n \times n$ identity matrix.

   b. Can we necessarily take $k = n$? Explain your answer.

   c. Does the result of part a still hold if we drop the separability assumption? Explain your answer.

8. Let F be a field and assume that every odd-degree polynomial in $F[x]$ has a root in F. Let $K \supseteq F$ be a finite extension such that $[K : F] > 1$. Prove that the degree $[K : F]$ is even.

9. Let $K \supseteq F$ be a Galois extension of degree 25, and assume that there are two distinct intermediate fields $F \subsetneq E_1, E_2 \subsetneq K$ lying strictly between F and K. How many total

intermediate fields lie strictly between F and K? Explain your answer. (Results from Galois theory, group theory, or linear algebra may be used without proof.)

10. Let $K = F(\alpha)$ be a finite Galois extension of F, with $\alpha \notin F$ and with $\alpha \neq \alpha^{-1}$. Assume there exists $\sigma \in \mathrm{Gal}(K/F)$ such that $\sigma(\alpha) = \alpha^{-1}$. Prove that $[K : F(\alpha+\alpha^{-1})] = 2$. (Hint. One approach is to prove both $[K : F(\alpha + \alpha^{-1})] \leq 2$ and $[K : F(\alpha + \alpha^{-1})] \geq 2$.)