ALGEBRA COMPREHENSIVE EXAM

Monday, 17 June 2019

Math Exam ID#: ____________________

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		20
Total		100

Instructions: Justify your answers. Clearly indicate your final answers, and cross out work that you do not wish to be considered. The first eight questions are worth 10 points, and the True/False question is worth 20 points. Do as many problems as you can, as completely as you can. The exam is two and one-half hours. No notes, books, or calculators.

1. Let p, q denote distinct primes. Assume G is a finite group, and assume that G has a unique Sylow p-subgroup and also a unique Sylow q-subgroup. Assume $g_1 \in G$ has order p and $g_2 \in G$ has order q. Prove that $g_1 g_2 = g_2 g_1$. (Possible hint. Prove that $g_1 g_2 g_1^{-1} g_2^{-1}$ is trivial.)
2. Let G be a finite simple group having a subgroup H of prime index p. Prove that p is the largest prime divisor of $|G|$.
3. Suppose I is an ideal of $R = \mathbb{Z}[x]$ and suppose $p \in I$ for some prime number p. Prove that I can be generated by 2 elements.
4. Let $\mathfrak{p}$ denote a nonzero prime ideal in the ring $\mathbb{Z}[\sqrt{3}] = \{a + b\sqrt{3} \colon a, b \in \mathbb{Z}\}$. Prove that $\mathfrak{p}$ contains a unique prime integer. (For example, the prime ideal $(1 + \sqrt{3})$ contains the prime integer 2.)
5. Let R be the ring of $n \times n$ matrices over a field F. Prove that every nonzero element of R is either a unit or a zero-divisor.
6. Let A, B be two $n \times n$ complex matrices, and assume $AB = BA$. Prove that A and B have a common eigenvector.
7. Suppose A is an $n \times n$ matrix over a field K with minimal polynomial $m(x)$. Let $f(x) \in K[x]$ be a polynomial. Prove that $f(A)$ is nonsingular if and only if $f(x)$ and $m(x)$ are relatively prime in $K[x]$.
8. Let $f(x) \in \mathbb{Q}[x]$ be irreducible of odd degree. If α is a root of $f(x)$, prove that $\mathbb{Q}(\alpha) = \mathbb{Q}(\alpha^{2^k})$, for all integers $k > 0$.
9. (20 points; 5 points each.) Answer True or False to the following questions and briefly explain your answers.
   a. The polynomial $x^4 + 1$ is irreducible over $\mathbb{R}$.
   b. If n divides the order of a finite group G, then there exists $g \in G$ of order n.
   c. The intersection of two prime ideals is a prime ideal.
   d. If the group G acts on the set S, and if $s \in S$ is an element, then the stabilizer of s is a normal subgroup of G.