ALGEBRA COMPREHENSIVE EXAM

Monday, 18 June 2018

Math Exam ID#: ____________________

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total		100

Instructions: Justify your answers. Clearly indicate your final answers, and cross out work that you do not wish to be considered. Each of the 10 questions is worth 10 points. Do as many problems as you can, as completely as you can. The exam is two and one-half hours. No notes, books, or calculators.

1. a. For which integers $n \geq 2$ is the group $\{(1), (12)\}$ a normal subgroup of the symmetric group S_n? Prove your answer.

   b. Assume G is a group and N is a normal subgroup of G such that both N and G/N are cyclic. Prove that G can be generated by two elements.

2. Let G denote a finite group and let $H \leq G$ be a subgroup of index 2.

   a. Prove that if $x \in G$ and $x \notin H$, then x has even order in G.

   b. If $x \in H$, does x necessarily have even order in G? Prove your answer.

3. Describe, up to isomorphism, all finitely generated abelian groups G such that the map $g \mapsto g + g$ is surjective. Explain your answer.

4. The group $\mathrm{GL}_2(\mathbb{C})$ acts on the set $\mathbb{C}^{2\times 2}$ of 2×2 complex matrices by conjugation. Classify the orbits of this action. (For example, you could give a list of representatives for the orbits, with one unique representative for each orbit.)

5. Assume V is an n-dimensional vector space and $T : V \to V$ is a linear transformation. Assume $T^m = 0$ for some integer $m \geq 1$. Prove that $T^n = 0$.

6. Prove or give a counter-example: If U_1, U_2 are subspaces of a vector space V and if $v_1, v_2 \in V$ are such that
$$v_1 + U_1 = v_2 + U_2,$$
then $U_1 = U_2$.

7. Let $T : V \to V$ denote a linear transformation of a vector space V and let $v_1, \ldots, v_n \in V$ denote eigenvectors of T with pairwise distinct eigenvalues. Prove that the vectors $v_1, \ldots, v_n$ are linearly independent.

8. For each of the following ideals in $\mathbb{F}_3[x, y]$ determine whether the ideal is prime and whether it is maximal. You don't need to justify your answers.
(i): (y) (ii): $(2x + y, 2)$ (iii): $(2x + y, x - 2y)$ (iv): $(x^2 + y^2, x + y)$

9. Let R be a commutative ring with unity $1 \neq 0$.

   a. Assume R has precisely one maximal ideal $\mathcal{I}$. Prove that every $r \in R \setminus \mathcal{I}$ is a unit in R.

   b. Assume the set of all non-unit elements of R constitute an ideal; denote this ideal by $\mathcal{I}$. Prove that $\mathcal{I}$ is a maximal ideal in R, and it is the only maximal ideal in R.

10. Consider the polynomial
$$p(x) = x^3 + ax + 1$$
in $\mathbb{F}_3[x]$. You don't need to justify your answers.

    a. For which values $a \in \mathbb{F}_3$ does $p(x)$ have at least one root in $\mathbb{F}_9$?

    b. For which values $a \in \mathbb{F}_3$ does $p(x)$ have all its roots in $\mathbb{F}_9$?