ALGEBRA QUALIFYING EXAM
JUNE 2024

Math Exam ID#: ____________________

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total		100

Instructions: All problems are worth 10 points. Unless otherwise specified, all rings are commutative, have unity, are not the zero ring, and all ring homomorphisms map unity to unity. Write your proofs clearly using complete sentences. Your proofs will be graded based on clarity as well as correctness; a correct answer will not receive full credit if the reasoning is difficult to follow. Good luck.

1. a. Prove that every group of order 15 is cyclic.

   b. Is every group of order 21 cyclic? Either prove that the answer is 'Yes' or give an example to show that the answer is 'No'.

2. a. Prove that if G is a group of order 12, and $H \leq G$ is a subgroup of order 6, then H contains g^2 for every element $g \in G$.
   (Here we are assuming the group operation is written using multiplication.)

   b. Prove that the alternating group A_4 has no subgroup of order 6.
   (You may use the previous part, or you may prove this using a different argument.)

3. Prove that if H has finite index n in G, then there exists a normal subgroup K of G with $K \leq H$ and $[G : K] \leq n!$ (the factorial of n). Do not assume that G is finite.

4. Is $\mathbb{Z}[\frac{1}{2}]$ finitely generated as a $\mathbb{Z}$-module? Prove your answer.
   Reminder/notation: $\mathbb{Z}[\frac{1}{2}]$ is the subring of $\mathbb{Q}$ given by
   $$\left\{\frac{a}{b} \in \mathbb{Q} \colon a, b \in \mathbb{Z}, b = 2^m \text{ for some } m \geq 0\right\}.$$

5. Let R be a Noetherian commutative ring, let I, $\mathfrak{p}$ be ideals in R with $\mathfrak{p}$ a prime ideal, and assume that $I \subseteq \mathfrak{p}$ and, moreover, that $\mathfrak{p}$ is the *only* prime ideal of R containing I. Prove that there exists an integer $m > 0$ such that $\mathfrak{p}^m \subseteq I$.
   (Hint. You may use without proof that in a commutative ring that is not the zero ring, the nilpotent elements are precisely the intersection of all prime ideals in that ring.)

6. Let F be a field, and let I_n denote the $n \times n$ identity matrix in $M_{n\times n}(F)$.
   Assume $A \in M_{n\times n}(F)$ satisfies $A^2 = I_n$ and $A \neq I_n$. Prove that -1 is an eigenvalue of A.

7. For which positive integers e is the polynomial $x^2 + x + 1$ irreducible over $\mathbb{F}_{2^e}$?
   Explain your answer.

8. Assume $K \supseteq F$ is a (finite) Galois extension. Assume $\alpha \in K$ and $\beta \in F \setminus \{0\}$ are such that $\sigma(\alpha) = \alpha + \beta$ for some $\sigma \in \mathrm{Gal}(K/F)$. Prove that K has positive characteristic.

9. In this problem, you may use without proof that $\mathbb{Q}(\sqrt{3 + 2\sqrt{2}})$ is degree four over $\mathbb{Q}$.

   a. Prove that $\mathbb{Q}(\sqrt{3 + 2\sqrt{2}})$ is Galois over $\mathbb{Q}$.

   b. Compute the Galois group of $\mathbb{Q}(\sqrt{3 + 2\sqrt{2}})$ over $\mathbb{Q}$. Justify your answer.

10. Give an example or state that no such example exists. Provide a brief explanation in either case.

    a. A UFD which is not a Euclidean Domain.

    b. A positive integer n such that $\mathbb{Q}(\zeta_n)$ contains two distinct quadratic (degree 2) extensions of $\mathbb{Q}$.