ALGEBRA QUALIFYING EXAM SPRING 2019 QUESTIONS

1. Does the symmetric group S_5 contain a subgroup isomorphic to:

(a) The dihedral group D_8 with 8 elements?
(b) The quaternion group Q_8 with 8 elements?

2. Suppose A is a finitely generated abelian group, B is a subgroup of A, and $C = A/B$. Prove that if C is torsion free then the isomorphism classes of B and C determine the isomorphism class of A uniquely.

Give a counterexample that shows the isomorphism class of A may not be uniquely determined if C has a non-trivial torsion.

3. For a group G, let $G_1 := G$ and let $G_{n+1} := [G, G_n]$. We say G is nilpotent if $G_N = 1$ for some N. Prove that if G is a p-group, i.e. $|G| = p^r$ for some prime p, then G is nilpotent.

(Recall that if H, K are subgroups of G then $[H, K] = \langle [h, k] \mid h \in H \text{ and } k \in K \rangle$ where $[h, k] = h^{-1}k^{-1}hk$.)

4. Throughout this question we assume that R is a commutative ring with 1.

(a) Let A be a multiplicative subset of R (that is, $0 \notin A$ and $ab \in A$ whenever $a, b \in A$). Consider an ideal P of R such that $P \cap A = \varnothing$, and P is maximal with this property (that is, $P' \cap A \neq \varnothing$ whenever $P' \supsetneq P$ is an ideal of R). Prove that P is a prime ideal of R.
(b) Recall that

$$\mathfrak{N}(R) = \{r \in R \mid r^n = 0 \text{ for some } n \in \mathbb{Z}^+\}$$

is an ideal of R, called the nilradical of R. (Do not prove that $\mathfrak{N}(R)$ is an ideal!). Prove that the following are equivalent:
(i) R has exactly one prime ideal.
(ii) Every element of R is either nilpotent (that is, an element of $\mathfrak{N}(R)$) or a unit.
(iii) $R/\mathfrak{N}(R)$ is a field.

5. Recall that the ring $\mathbb{Z}[i]$ of Gaussian integers has a Euclidean norm.

(a) Prove that for every ideal $I \neq 0$ of $\mathbb{Z}[i]$, the quotient $\mathbb{Z}[i]/I$ is a finite ring.
(b) Identify what is $\mathbb{Z}[i]/(1 + i)$.

6. Assume n is a squarefree integer, i.e., n is a product of distinct primes. Prove that the primitive n-th roots of unity constitute a basis of the cyclotomic field $\mathbb{Q}(\zeta_n)$ over $\mathbb{Q}$. (Here "basis" is meant in the sense of vector spaces.)

7. Calculate the number of primitive elements of $\mathbb{F}_{27}$ over $\mathbb{F}_3$. Recall that if K/F is a field extension then $\alpha \in K$ is called a primitive element of K over F if and only if $K = F(\alpha)$.

8. Let K/F be a Galois algebraic extension with no proper intermediate fields. Prove that $[K : F]$ is prime.

9. Let V be a vector space of over the field $\mathbb{Q}$ of rational numbers of dimension at most $p-2$ where p is a prime. Let T be a linear operator on V such that $T^p = I$. Show that $T = I$.

10. Consider $n \times n$ matrices A, B over $\mathbb{C}$ such that the following are satisfied:

(a) A, B are nilpotent with the same nilpotency index (recall the nilpotency index of a matrix X is the smallest number k such that $X^k = 0$).

(b) $\mathsf{rank}(A) = \mathsf{rank}(B)$.

(c) $\mathsf{rank}(A^2) = \mathsf{rank}(B^2)$.

Prove the following:

(i) If $n > 9$ then A, B may be non-similar.

(ii) If $n \leq 9$ then A, B are similar.