MATH EXAM ID#________________________

Algebra Qualifying Exam Fall 2022

Total Points Possible	
Question	Score
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Qualifying Exam in Algebra

Department of Mathematics, UC Irvine

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1. Let G be a group, X be the set of all homomorphisms $h : G \to G$ and Y be the set of all automorphisms of G. Consider the group actions of G on X, Y defined by $a \cdot h = h \circ f_a$ where f_a is the inner automorphism $f_a(x) = axa^{-1}$.
   (a) For the action on X: Let $h \in X$. Determine the relationship between the stabilizer of h under this action and the centralizer $C_G(\mathrm{Im}(h))$.
   (b) For the action on Y: Prove that this action is transitive if and only if all automorphisms of G are inner.
2. Let A be a finite abelian group. Show that A is cyclic if and only if for any prime p it has either 0 or $(p-1)$ elements of order p.
3. Let $\mathbb{F}_p$ denote the field with p elements. Find the order of a Sylow p-subgroup of the group $G = GL_3(\mathbb{F}_p)$ and give an example of a Sylow p-subgroup.
4. Let R be a commutative ring with identity and $I \subset R$ an ideal such that $R/I \simeq \mathbb{Z}$. Show that there are infinitely many maximal ideals in R that contain I.
5. Let R be a unique factorization domain in which there exists a unique irreducible element up to associates.
   (a) Prove that R is a principal ideal domain.
   (b) Show that R has a unique maximal ideal and a unique non-maximal prime ideal.
6. Let $K \subset F$ be a submodule in a module over a commutative ring R which is a PID. Assuming that F is free of finite rank, show that K is also free of finite rank.
7. Consider the ring $R = \mathbb{Z}[x]$ and the ideal $I = (2, x)$ of R generated by elements 2 and x. Notice that the rings R/I and $\mathbb{Z}/2\mathbb{Z}$ are isomorphic.
   (a) Prove that the map $\varphi : I \times I \to \mathbb{Z}/2\mathbb{Z}$ defined by
   $$\varphi(a_0 + a_1 x + \cdots + a_n x^n, b_0 + b_1 x + \cdots + b_m x^m) = \frac{a_0}{2} b_1 \mod 2$$
   is R-balanced (i.e. it is additive in each component, and $\varphi(m, rn) = \varphi(mr, n)$ for all $m, n \in I$ and $r \in R$.)
   (b) Prove that the element $2 \otimes x - x \otimes 2 \neq 0$ in $I \otimes_R I$.

(c) Prove that the element $2 \otimes x - x \otimes 2$ is annihilated by both 2 and x in $I \otimes_R I$.

(d) Prove that the submodule of $I \otimes_R I$ generated by the element $2 \otimes x - x \otimes 2$ is isomorphic to R/I.

8. Assume F is a field and $F(\alpha)$ is a finite field extension of F of odd degree. Prove that $F(\alpha^2) = F(\alpha)$.

9. Let $F \supset \mathbb{Z}_2$ be a field with 64 elements and $\alpha \in F^* = F \setminus \{0\}$. Suppose that α has order $d \geq 8$ in the multiplicative group F^*. Show that α generates the field F over $\mathbb{Z}_2$, i.e. $\mathbb{Z}_2(\alpha) = F$.

10. Let $\mathbb{Q} \subset K$ be a Galois extension with Galois group isomorphic to S_5 and $L = K^C$ the fixed point subfield of the cyclic subgroup $C \subset S_5$ generated by the 5-cycle $\tau = (12345)$. Find the number of different subfields L' in K which are isomorphic to L (including the case $L' = L$).