ALGEBRA QUALIFYING EXAM: SOLUTIONS
JANUARY 9, 2023

Math Exam ID#: ____________________

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total		100

Instructions: All problems are worth 10 points. Unless otherwise specified, all rings are commutative, have unity and are not the zero ring. Write your proofs clearly using complete sentences. Your proofs will be graded based on clarity as well as correctness; a correct answer will not receive full credit if the reasoning is difficult to follow. Good luck.

1. a. Let G be a group and $Z(G)$ be the center of G.
   Prove that if $G/Z(G)$ is cyclic, then G is abelian.

   b. Let p be a prime. Prove that every group of order p^2 is abelian.

   c. Is it true that every group of order p^3 is abelian?
   Give a brief justification of your answer.

2. a. Let R be an integral domain. Define what it means for R to be a Euclidean domain.

   b. Let F be a field. Prove that $F[x]$ is a Euclidean domain.

3. Let G be a group of order $224 = 2^5 \cdot 7$. Prove that G has a subgroup of order 28.

4. Let $R = \mathbb{Z}[\sqrt{-3}]$ and let $S = \mathbb{Z}[i]$. Prove that there is no ring homomorphism $\varphi\colon R \to S$ that takes the (multiplicative) identity of R to the (multiplicative) identity of S.

5. Let R be a commutative ring with identity $1 \neq 0$. A module N over a ring R is *simple* if it is nonzero and it has no proper nonzero submodules. Prove that an R-module N is simple if and only if $N \cong R/M$ for a maximal ideal M of R.

6. Let k be an algebraically closed field, and fix distinct $\alpha, \beta \in k$. List all possible minimal polynomials $p(x) \in k[x]$ of 3×3 matrices over k whose eigenvalues are exactly α and β, and give an example of a matrix having each given $p(x)$ as its minimal polynomial.

7. Find the smallest Galois extension K of $\mathbb{Q}$ that contains $\mathbb{Q}(\sqrt[3]{5})$, and explain your answer.

8. Assume F is a field of characteristic 0 and α is such that the extension $F(\alpha)/F$ is finite of degree not divisible by 3. Is it true that $F(\alpha^3) = F(\alpha)$? Explain your answer.

9. Consider a finite extension of fields K/F. Prove that the following are equivalent.
   (a) K is a splitting field of some polynomial in $F[x]$ over F.
   (b) Every irreducible polynomial $p(x) \in F[x]$ which has a root in K splits in $K[x]$ into linear factors.

10. Recall that $\mathbb{Z}$ is the ring of integers, $\mathbb{R}$ is the ring of real numbers, and $\mathbb{C}$ is the ring of complex numbers. Also $i = \sqrt{-1}$ and $\mathbb{Z}[i]$ is the ring of Gaussian integers.
    Prove that $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathbb{C}$ are isomorphic as rings.
    Here recall that if S, S' are rings extending a ring R then on $S \otimes_R S'$ the ring multiplication is defined by letting $(s \otimes t)(s' \otimes t') = ss' \otimes tt'$ for all simple tensors.