Analysis Comprehensive Exam

June 18, 2019

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Problem 1: Let $G \subset \mathbb{R}^2$ be an open set and suppose that

$$[0,1] \times [0,1] \subseteq G.$$

Show that there exists $\epsilon > 0$ such that

$$[0,1+\epsilon] \times [0,1] \subseteq G.$$

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Problem 2: Let $\{f_n\}_{n\geq 1}$, and $\{g_n\}_{n\geq 1}$ be two sequences of functions defined on $[0, 1]$ such that f_n converges uniformly to f, and g_n converges uniformly to g on $[0, 1]$. Does it follow that $f_n g_n$ converges uniformly to fg? Explain your answer.

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Problem 3: A metric d on a space X is called an **ultrametric** if the triangle inequality is replaced by the following stronger property: for all $x, y, z \in X$, we have $d(x,z) \leq \max(d(x,y), d(y,z))$. Let (X,d) be an ultrametric space. Prove the following:

1. If B is an open ball in X, then any point in B is a center of B. (Recall that an open ball is a set of the form $B(x;r) := \{y \in X \ : d(x,y) < r\}$; x is referred to as a center of the ball.)

2. Every open ball in X is both open and closed.

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Problem 4: Let $E \subset \mathbb{R}$. Show that if every continuous function $f : E \to \mathbb{R}$ attains its maximum on E, i.e., $\sup_{x \in E} f(x) = f(a)$ for some $a \in E$, then E is compact.

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Problem 5: Suppose that $f : [0,1] \to \mathbb{R}$ is a function. Prove that f is continuously differentiable if and only if: for every $\epsilon > 0$, there are open intervals $I_1, \ldots, I_n$ such that $[0,1] \subseteq I_1 \cup \cdots \cup I_n$ and such that, for each $j = 1, \ldots, n$ and each $a, b, c, d \in I_j \cap [0,1]$ with $a \neq b$ and $c \neq d$, we have

$$\left| \frac{f(a) - f(b)}{a - b} - \frac{f(c) - f(d)}{c - d} \right| \leq \epsilon.$$

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Problem 6: Let $T : U \to V$ belong to $C^2(U)$, where U and V are open sets in $\mathbb{R}^2$. Assume the determinant of the matrix of first derivatives of T is the constant function 1. Denote the variables in U by (x_1, x_2) and the variables in V by (y_1, y_2). Recall that, for any differential form ω on V, ω_T denotes the differential form on U obtained by change of variables using T.

(a) Show that if $\omega = dy_1 \wedge dy_2$ then $\omega_T = dx_1 \wedge dx_2$.

(b) Let $\eta = y_1 dy_2$. Show that $d(x_1 dx_2 - \eta_T) = 0$.

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Problem 7: Let $f : [0,1] \to \mathbb{R}$ be continuous. Let $m := \min_{x\in[a,b]} f(x)$ and $M := \max_{x\in[a,b]} f(x)$. Show that for any $c \in [m, M]$, there exists a non-decreasing function α on [0,1], such that $\int_a^b f(x)d\alpha(x) = c$.

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Problem 8: Let $f : B \subset \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable map where B is the open unit ball in $\mathbb{R}^n$ centered at the origin. Suppose $\|\nabla f(x)\| \leq 1$ for all $x \in B$. Show that

$$|f(x) - f(y)| \leq \|x - y\|$$

for all $x, y \in B$.