Analysis Comprehensive Exam

June 18, 2019

Math Exam ID: ________________________________

SCORES:

1. ____________ /25
2. ____________ /25
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Problem 1: Let $G \subset \mathbb{R}^2$ be an open set and suppose that

$$[0, 1] \times [0, 1] \subseteq G.$$

Show that there exists $\epsilon > 0$ such that

$$[0, 1 + \epsilon] \times [0, 1] \subseteq G.$$
Problem 2: Let \( \{f_n\}_{n \geq 1} \) and \( \{g_n\}_{n \geq 1} \) be two sequences of functions defined on \([0, 1]\) such that \( f_n \) converges uniformly to \( f \), and \( g_n \) converges uniformly to \( g \) on \([0, 1]\). Does it follow that \( f_n g_n \) converges uniformly to \( fg \)? Explain your answer.
Problem 3: A metric $d$ on a space $X$ is called an ultrametric if the triangle inequality is replaced by the following stronger property: for all $x, y, z \in X$, we have $d(x, z) \leq \max(d(x, y), d(y, z))$. Let $(X, d)$ be an ultrametric space. Prove the following:

1. If $B$ is an open ball in $X$, then any point in $B$ is a center of $B$. (Recall that an open ball is a set of the form $B(x; r) := \{ y \in X : d(x, y) < r \}$; $x$ is referred to as a center of the ball.)

2. Every open ball in $X$ is both open and closed.
Problem 4: Let $E \subset \mathbb{R}$. Show that if every continuous function $f : E \to \mathbb{R}$ attains its maximum on $E$, i.e., $\sup_{x \in E} f(x) = f(a)$ for some $a \in E$, then $E$ is compact.
Problem 5: Suppose that $f : [0, 1] \to \mathbb{R}$ is a function. Prove that $f$ is continuously differentiable if and only if: for every $\epsilon > 0$, there are open intervals $I_1, \ldots, I_n$ such that $[0, 1] \subseteq I_1 \cup \cdots \cup I_n$ and such that, for each $j = 1, \ldots, n$ and each $a, b, c, d \in I_j \cap [0, 1]$ with $a \neq b$ and $c \neq d$, we have

$$\left| \frac{f(a) - f(b)}{a - b} - \frac{f(c) - f(d)}{c - d} \right| \leq \epsilon.$$
Problem 6: Let $T : U \to V$ belong to $C^2(U)$, where $U$ and $V$ are open sets in $\mathbb{R}^2$. Assume the determinant of the matrix of first derivatives of $T$ is the constant function 1. Denote the variables in $U$ by $(x_1, x_2)$ and the variables in $V$ by $(y_1, y_2)$. Recall that, for any differential form $\omega$ on $V$, $\omega_T$ denotes the differential form on $U$ obtained by change of variables using $T$.

(a) Show that if $\omega = dy_1 \wedge dy_2$ then $\omega_T = dx_1 \wedge dx_2$.

(b) Let $\eta = y_1 dy_2$. Show that $d(x_1 dx_2 - \eta_T) = 0$. 
Problem 7: Let $f : [0, 1] \to \mathbb{R}$ be continuous. Let $m := \min_{x \in [a,b]} f(x)$ and $M := \max_{x \in [a,b]} f(x)$. Show that for any $c \in [m, M]$, there exists a non-decreasing function $\alpha$ on $[0,1]$, such that $\int_a^b f(x) d\alpha(x) = c$. 
Problem 8: Let $f : B \subset \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable map where $B$ is the open unit ball in $\mathbb{R}^n$ centered at the origin. Suppose $\|\nabla f(x)\| \leq 1$ for all $x \in B$. Show that 
$$|f(x) - f(y)| \leq \|x - y\|$$
for all $x, y \in B$. 