Real Analysis Qualifying Exam
September 17, 2020

There are 6 problems on the exam. Do as many as you can.
Show your work and justify your answers. Write your answers on the page with the question and the blank page following.

Please turn off your cell phone and put it away.

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Problem 1: For $m \geq 1$, denote

$$\{q_m\}_{m\geq 1}$$

as the set of rational numbers in $[0,1]$. Denote

$$f_m(x) = |x - q_m|^{-\frac{3}{4}} \quad \text{for } x \in [0,1].$$

Prove that there exists a sequence of positive numbers $\{a_m\}_{m\geq 1}$ such that

$$\sum_{m=1}^{\infty} a_m f_m(x) \quad \text{converge for a.e } x \in [0,1]$$

Problem 2: f_n is a sequence of measurable functions on $[0,1]$. Prove that

$$\lim_{n\to+\infty}\int_{[0,1]}\frac{f_n}{1+|f_n|}\,dx=0$$

if and only if

$$f_n \text{ converges to 0 in measure}$$

Problem 3: Let $(X, \mathcal{M}, \mu)$ be a measure space. Let g be a real-valued $\mathcal{M}$-measurable function on X. Suppose fg is μ-integrable on X for every μ-integrable real-valued $\mathcal{M}$-measurable function f on X. Show that $\|g\|_\infty < \infty$.

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Problem 4: Let $(X, \mathcal{A}, \mu)$ be a measure space. Consider an $\mathcal{A}$-measurable function f on X such that $\|f\|_p < \infty$ for some $p \in (0, \infty)$. Show that

$$\lim_{N \to \infty} N^p \mu\{|f| \geq N\} = 0.$$

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Problem 5: A metric d on a space X is called an *ultrametric* if the triangle inequality is replaced by the following stronger property. For all $x, y, z \in X$:

$$d(x, z) \leq \max(d(x, y), d(y, z)).$$

Prove:

1. If B is an open ball in X, then any point in B is a center of B. (This means that for all $x \in B, B = B(x, r)$ for some r.)
2. Every open ball in X is both open and closed.

Problem 6: Compute

$$\lim_{n\to\infty} \int_0^1 \sin(nx)e^{-x^2}dx.$$

Be sure to justify your answer. (You may assume elementary facts about calculus, but state them.)