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Exam Time: 4:00pm-6:30pm, June 16, 2022

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Problem 1. Suppose $a_{m,n}$ is a positive number for each $m, n \in \mathbb{N}$. Is it true that

$$\limsup_{m\to\infty}\left(\limsup_{n\to\infty} a_{m,n}\right) = \limsup_{n\to\infty}\left(\limsup_{m\to\infty} a_{m,n}\right)?$$

Prove or give a counterexample.

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous positive function, and $F : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$F(x, y, z) = \Big(\int_0^x f(t)dt,\ \int_0^{x+y} f(t)dt,\ \int_0^{x+y+z} f(t)dt \Big).$$

Prove that F is locally (i.e. restricted to sufficiently small neighborhood of any point in $\mathbb{R}^3$) a C^1 diffeomorphism.

Problem 3. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuously differentiable functions on $[0,1]$ such that

$$|f_n'(x)| \leq x^{-\frac{1}{2022}} \quad \text{for} \quad x \neq 0 \text{ and } \int_0^1 f_n(x)dx = 2022$$

for each $n \in \mathbb{N}$. Prove that the sequence has a subsequence $\{f_{n_k}\}$ that converges uniformly on $[0,1]$.

Problem 4: Is $f(x) = \sum_{n=2}^{\infty} \left(\frac{x}{\ln n}\right)^n$ continuous on $(-\infty, +\infty)$? Explain.

Problem 5: Let X be a compact metric space, and suppose that the sequence $\{f_n\}$ in $C(X)$ decreases point-wise to a continuous function $f \in C(X)$; that is, $f_n(x) \geq f_{n+1}(x)$ for each n and x, and $f_n(x) \to f(x)$ for each x. Prove that the convergence is actually uniform on X.

Problem 6. Let $\mathbf{v}(x,y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}) : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$. Let

$$D = \{(x,y) : (x-2)^2 + y^2 < 9\}$$

be the disc in $\mathbb{R}^2$ centered at $(2,0)$ and radius 3. Let $\mathbf{n} = \mathbf{n}(x,y)$ be the unit outer normal vector to ∂D at $(x,y) \in \partial D$. Compute $\int_{\partial D} \mathbf{v} \cdot \mathbf{n} \, ds$.

Problem 7. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of real numbers such that

$$\lim_{n\to\infty} a_n = a, \quad \lim_{n\to\infty} b_n = b.$$

Prove

$$\lim_{n\to\infty} \frac{a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1}{n} = ab$$

Problem 8: Let f be a differentiable function on $\mathbb{R}$ such that

$$\lim_{|x|\to\infty} f(x) = 1.$$

Prove that there is a $x_0 \in \mathbb{R}$ such that $f'(x_0) = 0$.

9. Let $1 < p, q < \infty$ with $1/p + 1/q = 1$.

(i) Prove

$$xy \le \frac{x^p}{p} + \frac{y^q}{q}, \quad x, y > 0.$$

(ii) Let $f(x)$ and $g(x)$ be bounded Riemann integrable real-valued functions on the unit ball $B_n \subset \mathbb{R}^n$. Prove Hölder's inequality:

$$\Big| \int_{B_n} f(x)g(x)dx \Big| \le \Big(\int_{B_n} |f(x)|^p dx \Big)^{1/p} \Big(\int_{B_n} |g(x)|^q dx \Big)^{1/q}.$$